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A CLASS OF DISCONTINUOUS INTEGRALS

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1. Introduction. The main topic of this paper is the function $f(z)$ defined by the equation

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod\{\Gamma(\alpha_1+\alpha_2 s)\}}{\prod\{\Gamma(\beta_1+\beta_2 s)\}} z^{-s}\, ds, \tag{1'}$$

where each $\prod$ stands for the product of an arbitrary number of terms. We show that there are four distinct convergent types.

First type. The integral converges over a domain in the z-plane and defines an analytic function of z.

Second type. The integral converges for positive z only: it gives the values assumed for positive z by a function of z which is analytic in the domain $|\arg z| < \pi$ and $|z| > 0$.

Third type. The integral converges for positive z only: when z exceeds a certain number α, the integral gives the values assumed for positive z ($> \alpha$) by a function of z which is analytic in $|\arg z| < \pi$ and $|z| > \alpha$; when $0 < z < \alpha$, the integral gives the values assumed for positive z ($< \alpha$) by a function of z, distinct from the first function, analytic in $|\arg z| < \pi$ and $0 < |z| < \alpha$. For real values of z, $f(z)$ is continuous at $z = \alpha$.

Fourth type. When $|z| \neq \alpha$, the integral has the same general characteristics as the integral of the third type, but

(i) it does not converge, though it may have a principal value, when $z = \alpha$; (ii) $f(\alpha+0)$ and $f(\alpha-0)$ are in general both infinite; in the particular cases when both are finite the integral has a principal value $\frac{1}{2}\{f(\alpha+0)+\frac{1}{2}f(\alpha-0)\}$.

The application of these results to the real integral

$$I \equiv \int\limits_0^\infty f(t)f(at)t^{-\lambda}\, dt \qquad (a > 0) \tag{2'}$$

is of particular interest. If, for convenience, we write (1′) as

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)z^{-s}\, ds \tag{3'}$$

and then calculate I by means of Mellin transforms, the result is

$$\frac{a^{\lambda-1}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)\phi(1-\lambda-s)a^s\,ds. \tag{4'}$$

This integral is of the same general form as (3′); its type is determined by the type of (3′) and the value of λ.

Now it is the form of $\phi(s)$ which determines the type of the integral (3′), and the following facts, suggested in the first place by particular examples,* are intuitive, once (3′) has been examined in detail:

(i) if $\phi(s)$ gives an integral of the first type, then so does $\phi(s)\phi(1-s-\lambda)$, and I takes the values assumed for positive a by an analytic function of a;

(ii) if $\phi(s)$ gives an integral of the second type, then, leaving aside values of λ which lead to divergent integrals, $\phi(s)\phi(1-s-\lambda)$ gives an integral of the third or fourth type, according to the value of λ; I takes the values assumed for positive a by two distinct analytic functions of a, there being a circle $|a| = \alpha$ which separates the domains in which the functions correspond to I. The value of λ determines whether the numerical values of I are continuous at $a = \alpha$.

Thus, once the determining properties of the four types of (1′) have been obtained, we can see at once the inevitable character of the discontinuity of certain real integrals, such as the Weber-Schafheitlin integral† in the theory of Bessel functions. Moreover, we then have a whole class of such discontinuous integrals, particular members of which can be written down *ad libitum*.

2. Method of procedure. We consider the function $f(z)$ defined by

$$f(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)z^{-s}\,ds, \tag{1}$$

where (i) $\phi(s) = \Gamma(a+As)\Gamma(b-Bs)/\Gamma(c+Cs)\Gamma(d-Ds)$,

(ii) A, B, C, D denote positive numbers,

and (iii) $|\arg z| < \pi$.

Here $\phi(s)$ has two gamma functions in the numerator and two in the denominator. Other forms of $\phi(s)$, involving any number of

* The reader may see how the matter stands by considering the two cases (i) $\phi(s) = \Gamma(-s)$, (ii) $\phi(s) = \Gamma(-s)/\Gamma(s+\alpha)$.

† Cf. § 10, where this integral is given as an example.

gamma functions, present no special difficulty, as we use for the particular form only those methods which are capable of immediate extension to the more general form.

To avoid minor complications, we suppose that a, b, c, d denote real numbers and that the integral is taken along a straight-line path: the modifications necessary when a, b, c, d are not real or when the path is curved in some finite part of the s-plane are obvious.

Throughout, we suppose that no pole of the integrand lies on the path of integration; no further reference to this obvious proviso will be made.

3. Standard formulae. We write

$$s = \sigma + it = re^{i\theta}, \qquad z = Re^{i\phi},$$

so that, on using well-known results, we have the three formulae

$$|z^{-s}| = R^{-\sigma} e^{\phi t}; \tag{A}$$

$$|\Gamma(s)| = e^{-\frac{1}{2}\pi|t|} |t|^{\sigma-\frac{1}{2}} \sqrt{(2\pi)} \{1+o(1)\}, \tag{B}$$

where, as $|t| \to \infty$, $o(1) \to 0$ uniformly in any finite range of values of σ;

$$|\Gamma(s+\alpha)| = e^{-\sigma - t\theta} r^{\sigma+\alpha-\frac{1}{2}} \sqrt{(2\pi)} \{1+o(1)\}, \tag{C}$$

where, as $r \to \infty$, $o(1) \to 0$ uniformly in $|\theta| \leqslant \pi - \delta$ for any fixed small positive δ and for any fixed real constant α.

For convenience of writing, we use the notations

$$K = A+B-C-D, \tag{2}$$

$$\kappa = A-B-C+D, \tag{3}$$

$$k = (a-\tfrac{1}{2})+(b-\tfrac{1}{2})-(c-\tfrac{1}{2})-(d-\tfrac{1}{2}), \tag{4}$$

$$\mathfrak{F} = A^A B^{-B} / C^C D^{-D}. \tag{5}$$

If $K < 0$, then (1) diverges for all z. We therefore confine our attention to non-negative values of K.

4. First type*: $K = A+B-C-D > 0$.

For large values of $|t|$ the absolute value of the integrand of (1) is comparable with

$$e^{-\frac{1}{2}K\pi|t|} |t|^{\kappa\gamma+k} R^{-\gamma} e^{\phi t} \mathfrak{F}^{\gamma}.$$

It is readily proved that (1) defines an analytic function of z over any domain contained in $|\arg z| < \min(\pi, \tfrac{1}{2}K\pi)$. Here, and throughout, we exclude the particular point $z = 0$.

* The labels 'First type', 'Second type',... are in accordance with the classification in § 1. The justification of the label is the theme of the ensuing paragraph.

5. Second type:

$$K = A+B-C-D = 0, \qquad \kappa = A-B-C+D \neq 0.$$

5.1. For large values of $|t|$ the absolute value of the integrand of (1) is comparable with

$$|t|^{\kappa\gamma+k}(\mathfrak{F}/R)^{\gamma}e^{\phi t}, \tag{6}$$

and the integral is not convergent for any complex value of z. When z is positive, the details are much the same whether $\kappa > 0$ or $\kappa < 0$. We suppose that $\kappa < 0$.

When $K = 0$ and $\kappa < 0$, the integral (1) converges absolutely* for any positive z, if

$$\kappa\gamma+k < -1, \quad \text{i.e.,} \quad |\kappa|\gamma > 1+k. \tag{7}$$

As we have said, it does not converge for any complex value of z, but, as we shall show, there is an analytic function, defined over $|\arg z| < \pi$, whose values for positive z are given by (1), provided that γ satisfies condition (7).

5.11. We first choose a definite γ_1 to satisfy the two conditions

$$|\kappa|\gamma_1 \geqslant \tfrac{3}{2}+k, \qquad \gamma_1 > 0, \tag{8}$$

and then consider

$$\int \phi(s)R^{-s}\,ds, \tag{9}$$

along the path $\mathfrak{L}$, made up of

(i) $\mathfrak{L}_1$, the straight line from $\gamma+iT$ to γ_1+iT, where T is large and positive [if γ satisfies the conditions required of γ_1 we may take $\gamma_1 = \gamma$ and omit this part of the path];

(ii) $\mathfrak{L}_2$, that arc of the circle $|s| = r$ which lies to the right of $\sigma = \gamma_1$ and above $t = \sigma\tan\delta$, where $r^2 = \gamma_1^2+T^2$ and δ is any small fixed constant;

(iii) $\mathfrak{L}_3$, that arc of $|s| = r$ which lies below $t = \sigma\tan\delta$ and above $t = 1$.

5.12. We have, from (B), since $K = 0$,

$$\left|\int_{\mathfrak{L}_1}\right| = O\{T^{\kappa\sigma+k}(\mathfrak{F}/R)^{\sigma}\}$$

and, in virtue of (7), this is $o(1)$ as $T \to \infty$. On $\mathfrak{L}_2$, by (C),

$$|\Gamma(a+As)| \sim e^{-A\sigma-At\theta}(rA)^{A\sigma+a-\frac{1}{2}}\sqrt{(2\pi)}$$
$$= e^{-A\sigma+At(\frac{1}{2}\pi-\theta)}e^{-\frac{1}{2}A\pi t}(rA)^{A\sigma+a-\frac{1}{2}}\sqrt{(2\pi)}; \tag{10}$$

and so for $\Gamma(c+Cs)$.

* The integral converges for any positive z if $\kappa\gamma+k < 0$; this may be proved by extending $\mathfrak{L}_1$, in 5.11, to the left. Thus, when $K = 0$, $\kappa < 0$, a sufficient condition for the integral (1) to be of the second type is $|\kappa|\gamma > k$.

If we are to apply formula (C) to $\Gamma(b-Bs)$ when $\arg s = \theta$ and $\pi-\delta \geqslant \theta \geqslant \delta$, we must take $\arg(-Bs)$ to be $-\pi+\theta$. Doing this, we see that on $\mathfrak{L}_2$

$$|\Gamma(b-Bs)| \sim e^{B\sigma+Bt(-\pi+\theta)}(rB)^{-B\sigma+b-\frac{1}{2}}\sqrt{(2\pi)}$$
$$= e^{B\sigma-Bt(\frac{1}{2}\pi-\theta)}e^{-\frac{1}{2}Bt\pi}(rB)^{-B\sigma+b-\frac{1}{2}}\sqrt{(2\pi)}; \tag{11}$$

and so for $\Gamma(d-Ds)$.

Thus, on $\mathfrak{L}_2$, since $K = A+B-C-D = 0$,

$$|\phi(s)R^{-s}| = O\{e^{-\kappa\sigma+\kappa t(\frac{1}{2}\pi-\theta)}r^{\kappa\sigma+k}(\mathfrak{F}/R)^{\sigma}\}. \tag{12}$$

Now, on $\mathfrak{L}_2$, $\sigma = \gamma_1+\lambda$, where $\lambda \geqslant 0$, $\kappa < 0$, and $\kappa\gamma_1+k \leqslant -\frac{3}{2}$, so that

$$\left|\int_{\mathfrak{L}_2}\right| = O\{r^{-\frac{1}{2}}(r/e)^{\kappa\lambda}(\mathfrak{F}/R)^{\gamma_1+\lambda}\} \tag{13}$$

and this, for a fixed positive R, tends to zero as r tends to infinity.

5.13. To deal with $\mathfrak{L}_3$ we require a lemma. This lemma is of interest for other investigations: it runs

If $0 < \theta < \delta$ *but* $t \geqslant 1$, *then, instead of the asymptotic formula* (11), *which is valid when* $\theta \geqslant \delta$, *there is the following statement*:

$$|\Gamma(b-Bs)/e^{B\sigma+Bt(-\pi+\theta)}(rB)^{-B\sigma+b-\frac{1}{2}}|$$

has finite, positive, upper and lower bounds.

To prove this, we observe first that, when $t \geqslant 1$, then $Bt \geqslant B$ and the bounds of

$$|\sin\pi(b-Bs)|/e^{\pi B|t|}$$

are finite and positive. Moreover,

$$\Gamma(b-Bs) = \pi/\sin\pi(b-Bs)\Gamma(1-b+Bs);$$

and the statement of the lemma follows from (C).

5.14. It now follows that (13) holds when $\mathfrak{L}_3$ replaces $\mathfrak{L}_2$, and so we have

$$\int_{\mathfrak{L}}\phi(s)R^{-s}\,ds \to 0 \quad \text{as} \quad T \to \infty.$$

5.2. From the fact just established it follows that

$$\int_{\gamma+i}^{\gamma+i\infty} \phi(s)R^{-s}\,ds = \int_{\gamma+i}^{\infty+i} \phi(s)R^{-s}\,ds.$$

In a similar way it may be proved that

$$\int_{\gamma-i\infty}^{\gamma-i} \phi(s)R^{-s}\,ds = \int_{\infty-i}^{\gamma-i} \phi(s)R^{-s}\,ds,$$

so that we may replace (1), when $z = R > 0$, by

$$2\pi i f(R) = \int \phi(s) R^{-s}\, ds, \tag{14}$$

where the integral is taken along the three straight lines $\infty - i$ to $\gamma - i$; $\gamma - i$ to $\gamma + i$; $\gamma + i$ to $\infty + i$.

The integral (14) remains absolutely convergent when R is replaced by $z = Re^{i\phi}$ (for $|t| \leqslant 1$ throughout the path), and it is easily proved that (14), with z for R, gives a function of z analytic in any domain contained in

$$|\arg z| < \pi \quad \text{and} \quad |z| > 0.$$

We are not here concerned with 'completing the rectangle' of the path of (14) by crossing the real axis at suitable points and so evaluating (1) by the calculus of residues. This is easily done if A, B, C, D are rational.

6. Third type: $K = 0$, $\kappa = 0$, $k = a+b-c-d < -1$.

With these values of K, κ, k, it is clear, from (6), that the integral (1) will converge (absolutely) for any γ when z is positive.

On the circle $|s| = r$ we have, from (12),

$$|\phi(s) R^{-s}| = O\{r^k (\mathfrak{F}/R)^\sigma\} \tag{15}$$

and so,* when $R > \mathfrak{F}$

$$\int\limits_{\gamma+i}^{\gamma+i\infty} \phi(s) R^{-s}\, ds = \int\limits_{\gamma+i}^{\infty+i} \phi(s) R^{-s}\, ds.$$

We proceed as in 5.2 and so obtain a function of z, analytic in any domain contained in $|\arg z| < \pi$ and in $|z| > \mathfrak{F}$, whose values for positive z ($> \mathfrak{F}$) are given by (1).

When $R < \mathfrak{F}$ we use paths which lie to the left of $\sigma = \gamma$, and we then obtain a function of z, analytic in any domain contained in $|\arg z| < \pi$ and in $0 < |z| < \mathfrak{F}$, whose values for positive z ($< \mathfrak{F}$) are given by (1).

The two functions are not, in general, the same, but, over any range of positive values of z, and for any given γ, (1) is a continuous function of z. When $z = R$ and $|t|$ is large, the modulus of the integrand of (1) is a constant multiple of

$$|t|^k R^{-\gamma}\{1+o(1)\}$$

and, since $k < -1$, the continuity of the integral (1) is apparent.

* We omit the detailed discussion as being sufficiently obvious.

7. Fourth type: $K = 0$, $\kappa = 0$, $-1 \leqslant k < 0$.

7.1. There is now a break in the continuity of $f(z)$ for *real* values of z.

The absolute value of the integrand in (1) is still governed by the formula (15), which was obtained for the third type; the only difference is in the range of values of k under consideration.

7.11. Let R be fixed and greater than $\mathfrak{F}$.

We first choose any definite positive number γ_2 and a definite positive number λ such that $k+\lambda < 0$. We take T to be an arbitrary, large, positive number and put $r^2 = T^2+\gamma_2^2$. We then consider the integral (14) taken along the path l, made up of

(i) l_1, the straight line from $\gamma+iT$ to γ_2+iT;

[If $\gamma > 0$ we may take $\gamma_2 = \gamma$ and omit this part of the path.]

(ii) l_2, the arc of the circle $|s| = r$ between the point γ_2+iT and the point in the positive quadrant where the circle cuts the curve

$$\sigma = t^\lambda.$$

(iii) l_3, the arc of $|s| = r$ to the right of this curve and above $t = 1$.

By (15), we have

$$\left|\int_{l_1} \phi(s)R^{-s}\,ds\right| = O(T^k),$$

which tends to zero as T tends to infinity, since $k < 0$.

7.12. On l_2, $\sigma > 0$ and, from (15),

$$|\phi(s)R^{-s}| = O\{r^k(\mathfrak{F}/R)^\sigma\} = O(r^k). \tag{16}$$

As we shall show, the length of l_2 is $O(r^\lambda)$.

The point of intersection of the two curves

$$\sigma = t^\lambda, \qquad \sigma^2+t^2 = r^2$$

is given by

$$t^2+t^{2\lambda} = r^2. \tag{17}$$

In the positive quadrant $t > 0$, and it is clear, from (17), that $t < r$. Thus (17) gives,* since $\lambda > 0$,

$$r-t = \frac{t^{2\lambda}}{r+t} < \frac{t^{2\lambda}}{r} = O(r^{2\lambda-1}),$$

$$t = r\{1+O(r^{2\lambda-2})\}.$$

Hence, since $\lambda < |k| \leqslant 1$,

$$r-t = \frac{r^{2\lambda}\{1+O(r^{2\lambda-2})\}^{2\lambda}}{2r\{1+O(r^{2\lambda-2})\}} = \tfrac{1}{2}r^{2\lambda-1}\{1+O(r^{2\lambda-2})\}.$$

* We wish to thank the referee for pointing out this simple method of obtaining t in terms of r: our original method was less direct.

Accordingly, the point in the positive quadrant where the circle $|s| = r$ cuts the curve $\sigma = t^\lambda$ is given by

$$t = r\{1-\tfrac{1}{2}r^{2\lambda-2}+O(r^{4\lambda-4})\},$$
$$\sigma = r^\lambda\{1-\tfrac{1}{2}\lambda r^{2\lambda-2}+O(r^{4\lambda-4})\}.$$

It follows that the angle subtended at the centre of the circle $|s| = r$ by the arc l_2 is $O(r^{\lambda-1})$. Hence, by (16),

$$\left|\int\limits_{l_2} \phi(s)R^{-s}\,ds\right| = O(r^{k+\lambda}),$$

which tends to zero as r tends to infinity, since $k+\lambda < 0$.

7.13. On l_3, $$\sigma > r^\lambda\{1-\tfrac{1}{2}\lambda r^{2\lambda-2}+O(r^{4\lambda-4})\},$$

and so, when r is sufficiently large,

$$\sigma > \tfrac{1}{2}r^\lambda.$$

Since $\mathfrak{F} < R$, (15) now gives, in virtue of the inequality given above,

$$\left|\int\limits_{l_3} \phi(s)R^{-s}\,ds\right| = O\{r^{k+1}(\mathfrak{F}/R)^{\frac{1}{2}r^\lambda}\}$$

and this tends to zero as r tends to infinity.

7.2. Moreover, for $\sigma \geqslant 1$

$$|\phi(\sigma+i)R^{-\sigma+i}| = O\{\sigma^k(\mathfrak{F}/R)^\sigma\},$$

so that, since $\mathfrak{F} < R$, $$\int\limits_{\gamma+i}^{\infty+i} \phi(s)R^{-s}\,ds$$

is convergent. This fact, in conjunction with § 7.1, shows that

$$\int\limits_{\gamma+i}^{\gamma+i\infty} \phi(s)R^{-s}\,ds = \int\limits_{\gamma+i}^{\infty+i} \phi(s)R^{-s}\,ds,$$

and, in a similar way, it may be proved that

$$\int\limits_{\gamma-i\infty}^{\gamma-i} \phi(s)R^{-s}\,ds = \int\limits_{\infty-i}^{\gamma-i} \phi(s)R^{-s}\,ds.$$

Hence, when $z = R > \mathfrak{F}$, we may replace (1) by

$$2\pi i f(R) = \int \phi(s)R^{-s}\,ds,$$

where the integral is taken along the three straight lines $\infty-i$ to $\gamma-i$; $\gamma-i$ to $\gamma+i$; $\gamma+i$ to $\infty+i$.

As in § 6, we obtain a function of z, analytic in $|z| > \mathfrak{F}$ and $|\arg z| < \pi$, whose values for positive z ($> \mathfrak{F}$) are given by (1).

If, however, $R < \mathfrak{F}$, we use contours which lie to the left of $\sigma = \gamma$

and we then obtain a function of z, analytic in $0 < |z| < \mathfrak{F}$ and $|\arg z| < \pi$, whose values for positive z ($< \mathfrak{F}$) are given by (1).

7.3. In § 6, where $k < -1$, the two functions give rise to one continuous set of values for positive z. We shall see that this is no longer true when $-1 \leqslant k < 0$.

If, instead of using the formulae (B) and (C) for *absolute values*, we use the Stirling formula, then we readily find that, apart from factors which do not affect the convergence, the integrand of (1) is of the form

$$(\gamma+it)^k(\mathfrak{F}/R)^{it}$$

when $z = R > 0$ and $|t|$ is large.

Since $-1 \leqslant k < 0$, the integral will converge when $R \neq \mathfrak{F}$, but will not converge when $R = \mathfrak{F}$, so that there cannot be continuity as R passes through $\mathfrak{F}$.

7.4. *The nature of the discontinuity* at $R = \mathfrak{F}$ when $-1 < k < 0$.*

If $z = \mathfrak{F}$, then (1) is not convergent, though it may have a principal value. If it has a principal value, then, as λ tends to zero, the functions $f(\mathfrak{F}e^{\lambda})$ and $f(\mathfrak{F}e^{-\lambda})$ tend to infinity in such a way that

$$\lim_{\lambda\to 0}\{f(\mathfrak{F}e^{\lambda})+f(\mathfrak{F}e^{-\lambda})\}$$

is finite: the principal value is equal to this limit.

7.41. When α is fixed, $|\arg z| \leqslant \pi-\delta$ and $|\arg(z-\alpha)| \leqslant \pi-\delta$, the full Stirling formula† for the gamma function gives

$$\log\Gamma(z+\alpha) = (z+\alpha-\tfrac{1}{2})\log z-z+\tfrac{1}{2}\log(2\pi)+O(|z|^{-1}).$$

In applying this formula to $\phi(s)$ when $s = \gamma+it$, we first consider t to be positive. When $t > 0$, the substitutions $z = Ait$, $\alpha = A\gamma+a$ and $z = -Bit$, $\alpha = -B\gamma+b$ give, respectively,

$$\log\Gamma(a+A\gamma+Ait) = (Ait+A\gamma+a-\tfrac{1}{2})(\log At+\tfrac{1}{2}\pi i)-Ait+ \\ +\tfrac{1}{2}\log(2\pi)+O(t^{-1}),$$

$$\log\Gamma(b-B\gamma-Bit) = (-Bit-B\gamma+b-\tfrac{1}{2})(\log Bt-\tfrac{1}{2}\pi i)+Bit+ \\ +\tfrac{1}{2}\log(2\pi)+O(t^{-1}).$$

For convenience of writing, we introduce the notations

$$j = a-b-c+d, \qquad M = A^{a-\frac{1}{2}}B^{b-\frac{1}{2}}/C^{c-\frac{1}{2}}D^{d-\frac{1}{2}}.$$

* We are grateful to Professor Titchmarsh, whose work on a particular case we have found most helpful.

† E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1920), 278–9.

When $s = \gamma+it$ and $t > 0$, we have, remembering that $K = \kappa = 0$ (that is, $A = C$ and $B = D$) and $k = a+b-c-d$,

$$\log\{\Gamma(a+As)\Gamma(b-Bs)/\Gamma(c+Cs)\Gamma(d-Ds)\}$$
$$= k\log t+s\log \mathfrak{F} + \log M+\tfrac{1}{2}\pi ij+O(t^{-1}).$$

Accordingly, when $t > 0$,

$$\phi(\gamma+it)R^{-\gamma-it} = Mt^k(\mathfrak{F}/R)^{\gamma+it}e^{\frac{1}{2}\pi ij}\{1+O(t^{-1})\},$$
$$\phi(\gamma-it)R^{-\gamma+it} = Mt^k(\mathfrak{F}/R)^{\gamma-it}e^{-\frac{1}{2}\pi ij}\{1+O(t^{-1})\}.$$

7.42. We now put $(\mathfrak{F}/R) = \exp\lambda$ and consider the behaviour of (1) as $|\lambda| \to 0$. We use the notation $C_1(\lambda)$, $C_2(\lambda)$,... to denote functions of λ which are continuous at $\lambda = 0$.

When $\lambda \neq 0$, and T is any fixed positive number,

$$\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)R^{-s}\,ds$$
$$= \int\limits_{\gamma-iT}^{\gamma+iT} \phi(s)R^{-s}\,ds + \int\limits_{\gamma-i\infty}^{\gamma-iT} \phi(s)R^{-s}\,ds + \int\limits_{\gamma+iT}^{\gamma+i\infty} \phi(s)R^{-s}\,ds$$
$$= C_1(\lambda)+i\int\limits_T^\infty \{\phi(\gamma-it)R^{-\gamma+it}+\phi(\gamma+it)R^{-\gamma-it}\}\,dt,$$

or, on using the results of 7.41 and remembering that $k < 0$,

$$= C_1(\lambda)+C_2(\lambda)+Mi\int\limits_T^\infty t^k\{e^{\lambda(\gamma-it)}e^{-\frac{1}{2}\pi ij}+e^{\lambda(\gamma+it)}e^{\frac{1}{2}\pi ij}\}\,dt$$

$$= C_1(\lambda)+C_2(\lambda)+2Mie^{\lambda\gamma}\int\limits_T^\infty t^k\cos(\lambda t+\tfrac{1}{2}\pi j)\,dt.$$

7.43. Since $-1 < k < 0$, we have, replacing T by 0,

$$\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)R^{-s}\,ds = \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)e^{\lambda s}\mathfrak{F}^{-s}\,ds$$
$$= C_1(\lambda)+C_2(\lambda)+C_3(\lambda)+2Mie^{\lambda\gamma}\int\limits_0^\infty t^k\cos(\lambda t+\tfrac{1}{2}\pi j)\,dt.$$

Further, by a well-known result,

$$\int\limits_0^\infty t^k\cos(\lambda t+\tfrac{1}{2}\pi j)\,dt = \begin{cases} \dfrac{\Gamma(k+1)}{\lambda^{k+1}}\cos\tfrac{1}{2}(k+1+j)\pi & (\lambda > 0), \\[2ex] \dfrac{\Gamma(k+1)}{|\lambda|^{k+1}}\cos\tfrac{1}{2}(k+1-j)\pi & (\lambda < 0). \end{cases}$$

If $\frac{1}{2}(k+j)$ is an integer, that is to say, if $a-c$ is an integer, then $\cos\frac{1}{2}(k+1+j)\pi = 0$; if $\frac{1}{2}(k-j)$ is an integer, that is to say, if $b-d$ is

an integer, then $\cos\frac{1}{2}(k+1-j)\pi = 0$. Hence, remembering that $\mathfrak{F} > R$ implies $\lambda > 0$, we have the results

(i) $f(\mathfrak{F}-0)$ is finite only when $a-c$ is an integer (or zero); otherwise it is infinite:

(ii) $f(\mathfrak{F}+0)$ is finite only when $b-d$ is an integer (or zero); otherwise it is infinite.

It is not possible that both $f(\mathfrak{F}-0)$ and $f(\mathfrak{F}+0)$ be finite, since $(a-c)+(b-d) = k$, which is not an integer.

7.44. *The principal value.*

We use the notation of 7.42: e.g. $C_1(0)$ will mean the value assumed when $\lambda = 0$ by the function $C_1(\lambda)$ of 7.42.

Let T_1 be a variable number greater than T. Then

$$\int\limits_{\gamma-iT_1}^{\gamma+iT_1} \phi(s)\mathfrak{F}^{-s}\,ds = C_1(0)+i\int\limits_{T}^{T_1} \{\phi(\gamma-it)\mathfrak{F}^{-\gamma+it}+\phi(\gamma+it)\mathfrak{F}^{-\gamma-it}\}\,dt,$$

or, on using the results of 7.41,

$$= C_1(0)+C_2(0)+O(T_1^k)+Mi\int\limits_{T}^{T_1} t^k\{e^{-\frac{1}{2}\pi ij}+e^{\frac{1}{2}\pi ij}\}\,dt$$

$$= C_1(0)+C_2(0)+O(T_1^k)+C_3(0)+2Mi\int\limits_{0}^{T_1} t^k\cos\tfrac{1}{2}\pi j\,dt.$$

Hence, there is a principal value when $-1 < k < 0$ only if $\cos\frac{1}{2}\pi j = 0$. This principal value is then

$$C_1(0)+C_2(0)+C_3(0).$$

Since $\cos\frac{1}{2}\pi j = 0$ implies $\cos\frac{1}{2}(k+1+j)\pi = -\cos\frac{1}{2}(k+1-j)\pi$, and $-1 < k < 0$, it follows, from 7.43, that

$$\lim_{\lambda\to 0}\tfrac{1}{2}\{f(\mathfrak{F}e^{\lambda})+f(\mathfrak{F}e^{-\lambda})\} = \frac{1}{2\pi i}\{C_1(0)+C_2(0)+C_3(0)\},$$

which is the principal value of $f(\mathfrak{F})$.

7.5. *The nature of the discontinuity at $R = \mathfrak{F}$ when $k = -1$.*

If $z = \mathfrak{F}$, then (1) is not convergent, though it may have a principal value. If it has a principal value, then the limit-functions $f(\mathfrak{F}-0)$, $f(\mathfrak{F}+0)$ are finite and their mean is equal to the principal value. If (1) has not a principal value, then the limit-functions $f(\mathfrak{F}-0)$, $f(\mathfrak{F}+0)$ are not finite.

7.51. As in 7.42, we have, when $\lambda \neq 0$,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)R^{-s}\,ds = C_4(\lambda)+C_5(\lambda)+2Mie^{\lambda\gamma}\int_T^\infty t^{-1}\cos(\lambda t+\tfrac{1}{2}\pi j)\,dt,$$

where $C_4(\lambda)$, $C_5(\lambda)$ are the functions obtained by putting $k=-1$ in $C_1(\lambda)$, $C_2(\lambda)$.

7.52. Further,

$$\int_T^\infty t^{-1}\cos(\lambda t+\tfrac{1}{2}\pi j)\,dt = \cos\tfrac{1}{2}\pi j\int_{|\lambda|T}^\infty t^{-1}\cos t\,dt-(\operatorname{sgn}\lambda)\sin\tfrac{1}{2}\pi j\int_{|\lambda|T}^\infty t^{-1}\sin t\,dt.$$

If now j is an odd integer, $2m+1$ say, this reduces to

$$-(\operatorname{sgn}\lambda)(-1)^m\tfrac{1}{2}\pi+O(\lambda),$$

and so

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)R^{-s}\,ds = \int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)e^{\lambda s}\mathfrak{F}^{-s}\,ds$$
$$= C_4(\lambda)+C_5(\lambda)+Mie^{\lambda\gamma}\pi(-)^{m+1}(\operatorname{sgn}\lambda)+O(\lambda).$$

If j is not an odd integer, then, as $\lambda \to 0$,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)e^{\lambda s}\mathfrak{F}^{-s}\,ds$$
$$\sim 2Mie^{\lambda\gamma}\cos\tfrac{1}{2}\pi j\int_{|\lambda|T}^{1} t^{-1}\cos t\,dt \sim 2Mie^{\lambda\gamma}\cos\tfrac{1}{2}\pi j\log|\lambda^{-1}|.$$

We have thus proved that

(i) when $a-b-c+d$ is not an odd integer, both $f(\mathfrak{F}-0)$ and $f(\mathfrak{F}+0)$ are infinite;

(ii) when $a-b-c+d$ is an odd integer, both $f(\mathfrak{F}-0)$ and $f(\mathfrak{F}+0)$ are finite.

7.53. *The principal value.*

As in 7.44, we have

$$\int_{\gamma-iT_1}^{\gamma+iT_1} \phi(s)\mathfrak{F}^{-s}\,ds = C_4(0)+C_5(0)+O(T_1^{-1})+2Mi\cos\tfrac{1}{2}\pi j\int_T^{T_1} t^{-1}\,dt.$$

Hence, there is a principal value only when $\cos\frac{1}{2}\pi j = 0$, and its value is then $C_4(0)+C_5(0)$. It follows from 7.52 that

$$\tfrac{1}{2}\{f(\mathfrak{F}-0)+f(\mathfrak{F}+0)\} = \frac{1}{2\pi i}\{C_4(0)+C_5(0)\},$$

which is the principal value of $f(\mathfrak{F})$.

8. Extensions of the previous results. It is clear that the results remain true when some, but not all, of A, B, C, D are zero, provided that the definitions of K, κ, M, j, k are suitably modified: if, for example, $A = 0$, we merely delete the letter A and the term $(a-\frac{1}{2})$ from the definitions.

Further, if $\Gamma(a+As)$ is replaced by a product of n terms

$$\Gamma(a_1+A_1 s)...\Gamma(a_n+A_n s),$$

the results remain true provided that

(i) $\sum A_n$ replaces A in (2) and (3),

(ii) $\sum a_n - \frac{1}{2}n$ replaces $a-\frac{1}{2}$ in (4),

(iii) $\prod A_n^{A_n}$ replaces A^A in (5),

and, in 7.4 and 7.5,

(iv) $\sum a_n - \frac{1}{2}n$ replaces $a-\frac{1}{2}$ in the definition of j,

(v) $\prod A_n^{a_n-\frac{1}{2}}$ replaces $A^{a-\frac{1}{2}}$ in the definition of M.

In the same way the terms involving B, C, D may be replaced by products of gamma functions.

9. Integrals evaluated by Mellin's formula.

9.1. We now consider two pairs of Mellin's inversion formulae

$$\psi_n(x) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi_n(s)x^{-s}\,ds, \qquad \phi_n(s) = \int\limits_0^\infty x^{s-1}\psi_n(x)\,dx,$$

where $n = 1, 2$, and ϕ_1, ϕ_2 are combinations of gamma functions.

Suppose that values of γ and λ can be found such that the integral

$$\frac{1}{2\pi i}\int\limits_0^\infty \frac{\psi_1(a\theta)}{\theta^\lambda}\,d\theta \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi_2(s)\theta^{-s}\,ds, \tag{18}$$

converges absolutely when $a > 0$, and such that

$$\phi_1(1-\lambda-s) = \int\limits_0^\infty \theta^{-\lambda-s}\psi_1(\theta)\,d\theta, \tag{19}$$

$$2\pi i\psi_2(\theta) = \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \phi_2(s)\theta^{-s}\,ds. \tag{20}$$

The integral

$$\int\limits_0^\infty \psi_1(a\theta)\psi_2(\theta)\theta^{-\lambda}\,d\theta \qquad (a > 0) \tag{21}$$

is then given by (18), or, on changing the order of integration, by

$$\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\phi_2(s)\,ds\int\limits_0^\infty \psi_1(a\theta)\theta^{-\lambda-s}\,d\theta.$$

On using (19), this becomes

$$\frac{a^{\lambda-1}}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\phi_2(s)\phi_1(1-\lambda-s)a^s\,ds. \tag{22}$$

If now ϕ_1 and ϕ_2 are combinations of gamma functions such as we considered in §§ 2, 8, the following facts are in evidence. In the notation of §§ 3, 8, let

K_1, κ_1, k_1 refer to the combination $\phi_1(s)$,
K_2, κ_2, k_2 refer to the combination $\phi_2(s)$,
K, κ, k refer to $\phi_2(s)\ \phi_1(1-\lambda-s)$.

We then have

$$K = K_1+K_2, \qquad \kappa = \kappa_1-\kappa_2, \tag{23}$$

$$k = k_1+k_2+\kappa(1-\lambda). \tag{24}$$

We shall not attempt a classification of all possible cases, but confine ourselves to the two facts which show most clearly why certain integrals of the form (21) represent analytic functions of a, and others represent two distinct analytic functions of a over regions $|a| < \alpha$, $|a| > \alpha$, with or without continuity at $a = \alpha$ as the case may be.

We recall first that, for convergence of the Mellin integrals defining $\psi_n(x)$ $(n = 1, 2)$, we must have $K_1 \geqslant 0$, $K_2 \geqslant 0$. If either K_1 or K_2 is positive, then, by (23), K is positive and so (22) is of the first type (§ 4) and represents the values taken by an analytic function of a for positive values of a.

If $K_1 = K_2 = 0$ and, moreover, $\kappa_1 = \kappa_2$, then $K = 0$ and $\kappa = 0$. The resulting combination of (22), i.e.

$$\phi_2(s)\phi_1(1-\lambda-s)$$

is of the third type (§ 6) or the fourth type (§ 7) according as $k < -1$ or $-1 \leqslant k < 0$, i.e. according to the value of λ. In the one case $(k < -1)$, (22) represents for positive a two distinct functions of a analytic over regions separated by a circle $|a| = \alpha$, a certain constant, but is continuous for positive a at $a = \alpha$. In the other case $(-1 \leqslant k < 0)$ we get two distinct analytic functions which are not continuous at $a = \alpha$.

In the concluding section we give some illustrative examples.

9.2. The example worked in § 10 shows the limitations imposed when we use Mellin transforms as a method of proof in a particular case. The *form* of answer is shown by § 9, but, in this, the *proof* that (21) is given by (22) is dependent on the suppositions (18), (19), and (20). Other recent work provides instances* of this considerable power in indicating results and comparative lack of power in proving them on the part of Mellin transforms. The extension of Mellin's formula so that it can be made to prove the results it at present merely indicates seems to be one of the major problems of analytical technique.†

10. Illustrative examples.

(i) *The case* $K_1 > 0$, $K_2 = 0$. The Bessel functions $K_\nu(x)$, $J_\mu(x)$ can be represented,‡ respectively, as constant multiples of

$$\frac{1}{2\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty} \Gamma(-\nu-s)\Gamma(-s)(\tfrac{1}{2}x)^{\nu+2s}\,ds,$$

where $\gamma > 0$, $\gamma > R(\nu)$, and of

$$\frac{1}{2\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty} \frac{\Gamma(-s)}{\Gamma(\mu+s+1)}(\tfrac{1}{2}x)^{\mu+2s}\,ds,$$

where $\gamma > 0$ and $R(\mu) > 0$. The integral

$$\int_0^\infty K_\nu(ax)J_\mu(x)x^{-\lambda}\,dx$$

is a multiple of a hypergeometric function.§

(ii) *The case* $K_1 = K_2 = 0$, $\kappa_1 = \kappa_2$. We illustrate this case by showing how the transformations of § 9 apply to the proof of the Weber-Schafheitlin integral in the theory of Bessel functions.

If $R(\mu+\nu+1) > R(\lambda) > 0$ and $R(\mu) > 0$, the integral

$$\int_0^\infty J_\mu(a\theta)J_\nu(\theta)\theta^{-\lambda}\,d\theta$$

* e.g. G. N. Watson, *Proc. London Math. Soc.* (2) 35 (1933), 156–99; A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 6 (1935), 161–74 (162–3).

† Since writing this, we have learnt from Professor E. C. Titchmarsh that he has been working on such an extension of Mellin's formula.

‡ G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922), 192.

§ Watson, loc. cit. 410.

is convergent when $a > 0$ and is then equal to

$$\int_0^\infty \frac{J_\nu(\theta)}{\theta^\lambda}\frac{d\theta}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(-s)(\frac{1}{2}a\theta)^{\mu+2s}}{\Gamma(\mu+s+1)}\,ds, \tag{25}$$

where $\gamma < 0$.

For this to be absolutely convergent we require

at $\theta = 0$, $R(\lambda-\mu-\nu-2\gamma) < 1$ i.e. $2\gamma > -R(\mu+\nu+1-\lambda)$; (26)

at $\theta = \infty$, $R(\lambda-\mu-2\gamma+\frac{1}{2}) > 0$ i.e. $2\gamma < -R(\mu-\frac{1}{2}-\lambda)$. (27)

These can be satisfied if $R(\nu) > -\frac{3}{2}$. With $R(\nu) > -\frac{3}{2}$ and γ chosen to satisfy the above conditions, (25) may be written as

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(-s)(\frac{1}{2}a)^{\mu+2s}}{\Gamma(\mu+s+1)}\,ds\int_0^\infty \frac{J_\nu(\theta)\,d\theta}{\theta^{\lambda-\mu-2s}}.$$

The conditions (26) and (27) are precisely those required* to enable us to transform this repeated integral into

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(-s)\Gamma\{s+\frac{1}{2}(\mu+\nu+1-\lambda)\}a^{\mu+2s}}{\Gamma(\mu+s+1)\Gamma\{\nu+1-s-\frac{1}{2}(\mu+\nu+1-\lambda)\}2^\lambda}\,ds. \tag{28}$$

We do not attempt a complete discussion of the Weber-Schafheitlin integral from this standpoint, as we introduce it here merely to illustrate how the general theory works out in a particular case.

* Watson, loc. cit. 391, § 13.24 (1): this is the Mellin transform of the formula 192 (7).

PATH-SPACES OF HIGHER ORDER ✓

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IN the present note I deal with the differential geometry of the system of differential equations

$$\frac{d^{\sigma+1}}{dt^{\sigma+1}}x^i+\alpha^i\left(t,x,\frac{dx}{dt},...,\frac{d^\sigma x}{dt^\sigma}\right)=0 \quad (i=1,2,...,n;\sigma \geqslant 2). \qquad \text{(I)}$$

The curves that represent the solutions of this system will be called the *paths* of the space. The case $\sigma = 1$ is the most important in practice, as it includes Riemann space, Finsler space, and the usual path-spaces. Inasmuch as this, with the very simple case $\sigma = 0$, has been worked out elsewhere (**6**, **7**, **8**), it is assumed in all that follows that σ is greater than or equal to 2.

In order that a differential geometry may be associated with the paths, we shall assume (I) and its equations of variation to be tensor invariant under the transformation group

$$\bar{x}^i = F^i(x^1,x^2,...,x^n), \qquad \bar{t}=t \qquad \text{(A)}$$
$$F^i_j = \partial\bar{x}^i/\partial x^j, \qquad |F^i_j| \neq 0.$$

The principal differential operators and differential invariants of the space with respect to this group will be deduced. Existence theorems are omitted, as this note is concerned only with the formal apparatus of a differential geometry and a tensor calculus.

The calculus of variations gives an important special case of a *metric* geometry, the equations (I) then representing the extremals of some regular variational problem. The paths may be regarded as geodesics. A certain number of invariantive theories have been built up with this fact as a basis. That of De Donder (**3**) makes no attempt to discuss the geometrical apparatus for general metrics. Duschek and Mayer (**4**)* have gone one step beyond demonstrating the mere tensorial character of variational equations: they regard every fundamental equation or theorem in that subject as representing one in geometry though the correspondence is worked out only for $\sigma = 1$.

Further attempts have been made along the same lines for metrics of higher order. Kawaguchi (**5**), Craig (**1**, **2**), Synge (**9**) obtain several tensors and differential operators by means of various assumptions

* The statement of equivalence of variational and geometric phenomena occurs in one of Duschek's reviews in the *Zentralblatt für Math.*

and generalizations. But it is usually not clear just why those generalizations have been preferred to other possible methods. Some devices, such as the replacing of the metric $F\,dt$ by $F^2\,dt$ are not justified for the general F. It is also not obvious whether all the differential invariants and operators have been obtained.

Hilbert's fundamental theorem on invariants and its corollary, the theorem of Tresse on differential equations, lead us to expect that all differential invariants and operators of the space may be generated from a finite basis. In what follows I give the methods and the essential computations which derive this basis* for any system (I), whether metric or not.

1. We shall use the tensor-summation convention for repeated Latin indices, which are taken to range from 1 to n. Greek indices, or indices in brackets denote, in superscript, differentiation with respect to the parameter t of the corresponding order; in subscript, partial differentiation with respect to the various derivatives of x. Thus

$$u^{(0)i} = u^i, \qquad u^{\nu i} = u^{(\nu)i} = \frac{du^{(\nu-1)i}}{dt}, \tag{1.1}$$

$$f_{,i} = f_{(0)i} = \frac{\partial f}{\partial x^i}, \qquad f_{\nu i} = f_{(\nu)i} = \frac{\partial f}{\partial x^{\nu i}}.$$

Under the group (A), the derivatives of x transform as follows:

$$\begin{aligned}
\bar{x}^{(1)i} &= F^i_j x^{(1)j},\\
\bar{x}^{(2)i} &= F^i_r x^{(2)r} + F^i_{r,j} x^{(1)r} x^{(1)j},\\
\bar{x}^{(3)i} &= F^i_r x^{(3)r} + 3F^i_{r,j} x^{(2)r} x^{(1)j} + F^i_{r,j,k} x^{(1)r} x^{(1)j} x^{(1)k},\\
\bar{x}^{\nu i} &= F^i_r x^{\nu r} + \nu F^i_{r,j} x^{(\nu-1)r} x^{(1)j} + \psi[x, x^{(1)}, \dots, x^{(\nu-2)}].
\end{aligned} \tag{1.2}$$

The operator $\partial/\partial x^{\nu i}$ adds a covariant index to any tensor which does not contain derivatives of higher order than x^ν. In particular, $\partial/\partial x^{\sigma i}$ always does this, since derivatives of order $\sigma+1$ or more may be eliminated from any tensor by the use of (I). It is also obvious that

$$\partial \bar{x}^{\nu i}/\partial x^{\nu j} = F^i_j. \tag{1.3}$$

The α^i transform under (A) by the law

$$-\bar{\alpha}^i = -F^i_r \alpha^r + (\sigma+1) F^i_{r,j} x^{\sigma r} x^{(1)j} + \psi[x, x^{(1)}, \dots, x^{(\sigma-1)}]. \tag{1.4}$$

This shows, without investigating the structure of ψ, that $\partial\alpha^i/\partial t$ is a vector and that $\alpha^i_{\sigma j \sigma k}$ and further partial derivatives to x^σ are tensors of the rank indicated by the Latin indices.

* See (8) for a sketch of the method followed in the succeeding paragraphs.

For the remaining differential invariants, we shall first need an operator which corresponds to total differentiation with respect to along any fixed path, the *base* of the operation. We define this as

$$\frac{d}{dt} = -\alpha^r \frac{\partial}{\partial x^{\sigma r}} + \sum_{\nu=0}^{\sigma-1} x^{(\nu+1)r} \frac{\partial}{\partial x^{\nu r}} + \frac{\partial}{\partial t}. \tag{1.5}$$

This, however, is not tensorial in character. The general distributive operator which converts a tensor into another and corresponds to d/dt is known* to be of the form

$$\begin{aligned} \mathfrak{D} u^i &= du^i/dt + \gamma^i_r u^r, \\ \mathfrak{D} v_i &= dv_i/dt - \gamma^r_i v_r, \end{aligned} \tag{1.6}$$

for contravariant and covariant vectors respectively. For the general tensor, there will be a group of terms $\gamma^i_r T^{\cdot r \cdot}_{\dots}$ for each contravariant index, and another, $-\gamma^r_i T^{\dots}_{\cdot r \cdot}$ for each covariant index, as for covariant differentiation. Let it be noted that by these rules the operation $\mathfrak{D}$ may be performed for any geometric magnitude, tensorial or not. The coefficients of connexion γ^i_j have the law of transformation

$$\bar{\gamma}^i_k F^k_j + \frac{d}{dt} F^i_j = F^i_k \gamma^k_j. \tag{1.7}$$

Whereas these coefficients are otherwise completely arbitrary, the geometry for one set will differ from that for any other only with the occurrence of an additional invariant, the tensor $\gamma'^i_j - \gamma^i_j$. This is an *invariant of the connexion*. An intrinsic connexion and the associated intrinsic differential invariants remain to be deduced. It is again clear that $(\sigma+1)^{-1}\alpha^i_{\sigma j}$ have the proper law of transformation, and in the next section they will be shown to have a strong claim to the title of an intrinsic set of coefficients. Owing to the law of transformation (1.7), the coefficients of connexion may be made to vanish along any base by proper choice of coordinates. In that case the operator $\mathfrak{D}$ becomes precisely d/dt.

2. The connexion cannot be determined without recourse to the equations of variation of (I), which are

$$u^{(\sigma+1)i} + \sum_{\nu=0}^{\sigma} \alpha^i_{\nu r} u^{\nu r} = 0. \tag{II}$$

These are derived from (I) by the infinitesimal transformation $x'^i = x^i + u^i \delta t$ which carries paths into paths, u^i being a vector

* (6) 615.

variation. To reduce (II) to an invariantive form, we replace d/dt with $\mathfrak{D}$ as follows:

$$u^{(1)i} = \frac{du^i}{dt} = \mathfrak{D}u^i - \gamma^i_r u^r,$$
$$u^{(2)i} = \mathfrak{D}^2 u^i - 2\gamma^i_r \mathfrak{D}u^r + u^r\left\{-\frac{d}{dt}\gamma^i_r + \gamma^i_k \gamma^k_r\right\},$$
$$u^{(\nu)i} = \mathfrak{D}^\nu u^i + \sum_{\rho=1}^{\nu} \binom{\nu}{\rho} \mathfrak{D}^{\nu-\rho} u^r \theta^{\rho-1}(-\gamma^i_r), \tag{2.1}$$

where $\binom{\nu}{\rho}$ denotes a binomial coefficient.

The general $u^{\nu i}$ has been written down with the help of the symbolic operation θ defined by

$$\theta\beta^i_j = \frac{d}{dt}\beta^i_j - \gamma^r_j \beta^i_r. \tag{2.2}$$

We then have the tensorial form of (II),

$$\mathfrak{D}^{\sigma+1}u^i + \sum_{\nu=0}^{\sigma} \underset{\nu}{P^i_r} \mathfrak{D}^\nu u^r = 0, \tag{2.3}$$

where the coefficients $\underset{\nu}{P^i_r}$ are 'curvature tensors' of various orders, given by the formula

$$\underset{\rho}{P^i_j} = \binom{\sigma+1}{\rho} \theta^{\sigma-\rho}(-\gamma^i_j) + \sum_{\nu=0}^{\sigma} \binom{\nu}{\rho} \alpha^i_{\nu r} \theta^{\nu-\rho-1}(-\gamma^r_j). \tag{2.4}$$

The simplest of these is $\underset{\sigma}{P^i_j} = \alpha^i_{\sigma j} - (\sigma+1)\gamma^i_j$. The connexion is determined by putting this equal to zero, so that

$$\gamma^i_j = \frac{1}{\sigma+1} \alpha^i_{\sigma j}. \tag{2.5}$$

In all that follows we shall use this value, though the letter γ is retained for simplicity. The remaining σ tensors $\underset{0}{P^i_j}, \underset{(1)}{P^i_j}, \dots, \underset{(\sigma-1)}{P^i_j}$ are intrinsic differential invariants of the space, though not a complete set.

3. There exist other differential operators besides

$$\partial/\partial t, \qquad \mathfrak{D}, \qquad \underset{\sigma}{\nabla_i} = \partial/\partial x^{\sigma i}. \tag{3.1}$$

But they are all found by alternating these upon any tensor or vector.

The first of the three is of minor interest. The other two give

$$\underset{\sigma}{\nabla_j} \mathfrak{D}u^i - \mathfrak{D}\underset{\sigma}{\nabla_j} u^i = u^i_{(\sigma-1)j} - \sigma\gamma^r_j u^i_{\sigma r} + u^r \gamma^i_{r\sigma j}. \tag{3.2}$$

In this the last term $u^r\gamma^i_{r\sigma j}$ is tensorial and may be left out without loss of invariance. This gives another operator

$$\underset{(\sigma-1)}{\nabla_j}\, u^i = u^i_{(\sigma-1)j} - \sigma\gamma^r_j \underset{\sigma}{\nabla_r} u^i. \tag{3.3}$$

While $\underset{(\sigma-1)}{\nabla_j}$ adds a covariant index to any tensor, it does not correspond fully to covariant differentiation since there is no summation for each index. For any tensor $T^{\cdot\cdot}_{\cdot\cdot}$

$$\underset{(\sigma-1)}{\nabla_j}\, T^{\cdot\cdot}_{\cdot\cdot} = T^{\cdot\cdot}_{\cdot\cdot(\sigma-1)j} - \sigma\gamma^r_j \underset{\sigma}{\nabla_r} T^{\cdot\cdot}_{\cdot\cdot}. \tag{3.4}$$

Alternating $\underset{(\sigma-1)}{\nabla_j}$ and $\mathfrak{D}$, we obtain

$$\underset{(\sigma-1)}{\nabla_j}\mathfrak{D}u^i - \mathfrak{D}\underset{(\sigma-1)}{\nabla_j}\, u^i = u^i_{(\sigma-2)j} - (\sigma-1)\gamma^r_j \underset{(\sigma-1)}{\nabla_r}\, u^i + \overset{\sigma}{\underset{(\sigma-2)}{Q^r_j}} \underset{\sigma}{\nabla_r} u^i + u^r \underset{(\sigma-1)}{\nabla}\, \gamma^i_r. \tag{3.5}$$

Here again, the last term may be left out, $\underset{(\sigma-1)}{\nabla_r}\, \gamma^i_j$ being another differential invariant of the space if $\sigma > 2$. To see this, we may suppose the vector u^i to be free from $x^{(\sigma-2)}$, $x^{(\sigma-1)}$, x^σ. Then there is just one term left on the right-hand side, and that is tensorial for all vectors u^i. Thus a second operator of this new *secondary* type is obtained in the form

$$\underset{(\sigma-2)}{\nabla_j} = \frac{\partial}{\partial x^{(\sigma-2)j}} - (\sigma-1)\gamma^r_j \underset{(\sigma-1)}{\nabla_r} + \overset{\sigma}{\underset{(\sigma-2)}{Q^r_j}} \underset{\sigma}{\nabla_r}. \tag{3.6}$$

The process may be carried out again. We obtain a $\underset{\nu}{\nabla_j}$ defined by

$$\underset{\nu}{\nabla_j} = \frac{\partial}{\partial x^{\nu j}} + \sum_{\rho=\nu+1}^{\sigma} \overset{\rho}{\underset{\nu}{Q^r_j}} \underset{\rho}{\nabla_r}. \tag{3.7}$$

The coefficients are given by $\overset{\rho+1}{\underset{\rho}{Q^i_j}} = -(\rho+1)\gamma^i_j$; and

$$\overset{\rho}{\underset{\nu}{Q^i_j}} = \overset{\rho+1}{\underset{\nu+1}{Q^i_j}} - \left\{\frac{d}{dt} \overset{\rho}{\underset{\nu+1}{Q^i_j}} + \gamma^i_r \overset{\rho}{\underset{\nu+1}{Q^r_j}}\right\} \qquad (\nu+1 < \rho < \sigma), \tag{3.8 a}$$

$$\overset{\sigma}{\underset{\nu}{Q^i_j}} = -\underset{(\nu+1)}{\alpha^i_j} - \left\{\frac{d}{dt} \overset{\sigma}{\underset{\nu+1}{Q^i_j}} + \gamma^i_r \overset{\sigma}{\underset{\nu+1}{Q^r_j}}\right\} \qquad (\nu+1 < \sigma). \tag{3.8 b}$$

In the very last operator, however, the terms in u^r may not be omitted, for our argument cannot be applied to vectors for which $u^i_{,j} = 0$, since that would make u^i an identically constant vector—a property which certainly cannot hold for any choice of coordinates and for a non-vanishing vector. $\underset{0}{\nabla}$ will, therefore, be of the form

$$\underset{0}{\nabla_j} u^i = u^i_{(0)j} - \gamma^r_j \underset{1}{\nabla_r} u^i + \overset{2}{\underset{0}{Q^r_j}} \underset{2}{\nabla_r} u^i + \ldots + \overset{\sigma}{\underset{0}{Q^r_j}} \underset{\sigma}{\nabla_r} u^i + u^r \underset{1}{\nabla_j} \gamma^i_r. \tag{3.9}$$

This corresponds to the covariant derivative in that a summation with the proper sign will be present for each index, with $\underset{1}{\nabla_j}\gamma^i_r$ representing the Christoffel symbols of Riemannian geometry. But $\underset{2}{\nabla_j}\gamma^i_r, \underset{3}{\nabla_j}\gamma^i_r, \ldots, \underset{\sigma}{\nabla_j}\gamma^i_r$ are tensors.

This furnishes the basis for all operators of the intrinsic connexion. Further alternations only give extra differential invariants as coefficients on the right-hand side of each identity.

The invariants $\underset{\nu}{\nabla_k}\gamma^i_j$ are, except $\underset{\sigma}{\nabla_j}\gamma^i_k$, not independent of our primary curvature tensors. In fact, we have

$$\underset{(\sigma-1)}{P^i_j} = \alpha^i_{(\sigma-1)j} - \frac{\sigma(\sigma+1)}{2}\left\{\frac{d}{dt}\gamma^i_j + \gamma^i_r\gamma^r_j\right\}.$$

This gives, with very little calculation,

$$\underset{(\sigma-1)}{\nabla_j}\gamma^i_k = \gamma^i_{k(\sigma-1)j} - \sigma\gamma^i_{k\sigma r}\gamma^r_j = \frac{1}{4-\sigma^2}\left\{4 \underset{(\sigma-1)}{P^i_{j\sigma k}} + 2\sigma \underset{(\sigma-1)}{P^i_{k\sigma j}}\right\}. \quad (3.10)$$

Similarly for the others. If we alternate $\underset{0}{\nabla}$ and $\mathfrak{D}$, these primary tensors again appear as coefficients on the right-hand side. Alternating $\partial/\partial t$ with the other operators gives no new operators, but the essentially new differential invariant $\partial\alpha^i/\partial t$, with others that can be deduced from it.

4. The existence of a metric makes our path-geometry merely a formal apparatus for the calculus of variations in the small. A known necessary condition that a metric exist is that the equations of variation (II) be self-adjoint. It is, in all probability, sufficient also, but I shall be content merely with deducing the formal conditions, without considering the proof of sufficiency here. The adjoint to a contravariant set of linear equations like (II) is covariant, and we shall be unable to interpret the term self-adjoint without some method of associating vectors of these two types. This is most conveniently done by means of a covariant tensor f_{ij} of rank two.* All the conditions under discussion may be obtained by varying

$$f_{ij}\{x^{(\sigma+1)j} + \alpha^j\} = 0. \quad (4.1)$$

These can be worked out explicitly by means of a lemma:

* (7) 5.

LEMMA. *The linear differential equations*

$$\underset{0}{A_{ij}}u^j+\underset{1}{A_{ij}}u^{(1)j}+...+\underset{m}{A_{ij}}u^{(m)j}=0 \tag{4.2}$$

are self-adjoint, if and only if

$$\underset{m}{A_{ij}}=(-1)^m\underset{m}{A_{ji}}, \qquad \underset{m-1}{A_{ij}}=(-1)^{m-1}\underset{m-1}{A_{ji}}+(-1)^m m\frac{d}{dt}\underset{m}{A_{ji}}, \tag{4.3}$$

$$\underset{p}{A_{ij}}=\sum_{r=p}^{m}(-1)^r\binom{r}{p}\underset{r}{A_{ij}^{(r-p)}},$$

where m is not summed.

In the present case, the equivalent of (4.3) must hold identically in t, x, and all the derivatives of x. One condition that is immediately obvious is

$$f_{ij}=(-1)^{\sigma+1}f_{ji}. \tag{4.4}$$

The fundamental tensor is symmetric if $\sigma+1$ is even; antisymmetric in the other case. But for a regular problem $|f_{ij}|$ must not vanish. That means that the product $n[\sigma+1]$ must always be even. This, however, could have been guessed from the fact that no variational problem can lead to equations of odd total order. The conditions (4.3), when applied to the varied equations of (4.1) fall into two classes. The first will include those equations which are to be guessed at from the existence of a variational principle, and the second leads to relations between the curvature tensors; (4.4) is one of the first class. Others would be that f_{ij} contain no x^μ if $\mu > \frac{1}{2}(\sigma+1)$. Furthermore, if $\sigma+1$ be even, f_{ij} will be of the form $f_{\lambda i \lambda j}$, λ being $[\frac{1}{2}(\sigma+1)]$; in the other case, f_{ij} would be a curl, $\phi_{i\lambda j}-\phi_{j\lambda i}$.

Conditions of the second class are best deduced directly from (II) and (2.3). The adjoint to these may be written in the form (4.5)

$$(-1)^{\sigma+1}\mathfrak{D}^{\sigma+1}v_i+\sum_{\nu=0}^{\sigma-1}(-1)^\nu\mathfrak{D}^\nu\Big(\underset{\nu}{P_i^r}v_r\Big)=0. \tag{4.5}$$

Here v_i is a covariant vector. These equations are best obtained from (2.3) by choosing a set of coordinates for which $\gamma^i_j=0$ along the base. Then $\mathfrak{D}$ becomes d/dt. The adjoint is found by requiring that v_i be a vector integrating factor. To arrive at the general invariantive form, we replace d/dt by $\mathfrak{D}$, and obtain the result desired. 'Self-adjoint' would mean $v_i=f_{ir}u^r$, with u^i a solution of (2.3). That is, (2.3), when contracted with $(-1)^{\sigma+1}f_{ij}$, must represent the same equations as

$$(-1)^{\sigma+1}\mathfrak{D}^{\sigma+1}f_{ir}u^r+\sum_{\nu=0}^{\sigma-1}(-1)^\nu\mathfrak{D}^\nu\Big\{f_{rk}\underset{\nu}{P_i^r}u^k\Big\}=0. \tag{4.6}$$

But in these last equations the coefficient of $D^\sigma u^r$ is Df_{ir}, and this must vanish, giving us a very important fundamental equation

$$Df_{ij} = 0. \tag{4.7}$$

The other coefficients are then equated to give

$$\begin{aligned} f_{ir} \underset{(\sigma-1)}{P^r_k} - f_{rk} \underset{(\sigma-1)}{P^r_i} &= 0, \\ f_{ir} \underset{(\sigma-2)}{P^r_k} - f_{rk}\Big\{-\underset{(\sigma-2)}{P^r_i} + (\sigma-1)\mathfrak{D} \underset{(\sigma-1)}{P^r_i}\Big\} &= 0, \\ f_{ir} \underset{\nu}{P^r_k} - f_{rk} \sum_{\rho=\nu}^{\sigma-1} (-1)^{\sigma+1-\rho} \binom{\rho}{\nu} \mathfrak{D}^{\rho-\nu} \underset{\rho}{P^r_i} &= 0, \quad \text{etc.} \end{aligned} \tag{4.8}$$

It should be noted that an antisymmetric f_{ij} does not give a proper geometry as in the Riemannian case; the length of a vector defined as $f_{ij} u^i u^j$ will always be zero. We also note in passing that the Eulerian equations of the calculus of variations may be rewritten in an invariantive form, though not so simply as for the case $\sigma = 1$. In fact, we have

$$\begin{aligned} \delta_i f &\equiv f_{(0)i} - f^{(1)}_{(1)i} + f^{(2)}_{(2)i} - f^{(3)}_{(3)i} + \dots + (-1)^\lambda f^\lambda_{\lambda i} \quad (\lambda = [\tfrac{1}{2}(\sigma+1)]) \\ &= \underset{0}{\nabla_i} f - \mathfrak{D} \underset{1}{\nabla_i} f + \underset{(\sigma-1)}{P^r_i} \underset{2}{\nabla_r} f + \mathfrak{D}^2 \underset{2}{\nabla_i} f + \dots = 0. \end{aligned} \tag{4.9}$$

The operators used are $\mathfrak{D}$ and $\underset{0}{\nabla}, \dots, \underset{\lambda}{\nabla}$; the coefficients are formed out of the primary curvature tensors $\underset{r}{P^i_r}$ and their derivates of various orders.

Lastly, an arbitrary $\phi(f)$ when substituted for f in $\delta\,(\int f\,dt) = 0$ will not give the same extremals unless a first integral $f =$ constant is available for the paths; the condition $Df = 0$ has the same significance here as in the case $\sigma = 1$.

REFERENCES

1. H. V. Craig, *American J. of Math.* 57 (1935), 450–62.
2. ——, *Trans. American Math. Soc.* 33 (1931), 125–42.
3. Th. De Donder, *Théorie Invariantive du Calcul des Variations* (Paris, 1930).
4. A. Duschek and W. Mayer, *Lehrbuch der Differentialgeometrie*, 2 (Leipzig, 1930).
5. A. Kawaguchi, *Rend. del Circ. Mat. di Palermo* 56 (1932), 245–76.
6. D. D. Kosambi, *Math. Zeits.* 37 (1933), 608–22.
7. ——, *Quart. J. of Math.* (Oxford) 6 (1935), 1–12.
8. ——, 'An affine calculus of variations': *Proc. Indian Ac. of Sci.* A, 2 (1935).
9. J. L. Synge, *American J. of Math.* 57 (1935), 679–91.

SERIES OF HYPERGEOMETRIC TYPE WHICH ARE INFINITE IN BOTH DIRECTIONS

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1. Introduction

ABOUT thirty years ago Dougall* gave the formula

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)} = \frac{\pi^2\Gamma(c+d-a-b-1)}{\sin\pi a \sin\pi b\,\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)}, \tag{1.1}$$

where $R(c+d-a-b-1) > 0$. When $d = 1$, this formula reduces to Gauss's well-known result

$$F[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

If $(a)_{-n}$ is, as usual, interpreted as meaning $(-1)^n/(1-a)_n$ when n is a positive integer, (1.1) can be written in the form

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n(b)_n}{(c)_n(d)_n} = \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)}, \tag{1.2}$$

and the series on the left may be regarded as a hypergeometric series (with unit argument) which is infinite in both directions.

It is convenient to denote the series on the left of (1.2) by ${}_2H_2[a,b;c,d]$, so that (1.2) sums the general ${}_2H_2$ with unit argument, and can be written as

$${}_2H_2[a,b;c,d] = \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)}, \tag{1.3}$$

the argument in the series ${}_2H_2$ being omitted when it is $+1$, as in the case of generalized hypergeometric series.

More generally we define ${}_pH_p$ as

$${}_pH_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\,z\right] = \sum_{n=-\infty}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\rho_1)_n(\rho_2)_n\cdots(\rho_p)_n} z^n,$$

where, of course, $|z| = 1$.

For example,

$${}_1H_1[a;b] = \ldots+\frac{(b-1)(b-2)}{(a-1)(a-2)}+\frac{b-1}{a-1}+1+\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\ldots.$$

* Dougall (2), § 13.

The bilateral series by which ${}_pH_p$ is defined can be separated into two parts, one consisting of the terms for which n is positive or zero, and the other containing the terms for which n is negative. Thus

$$ {}_pH_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\right] = {}_{p+1}F_p\left[\begin{matrix}1,\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\right]+ $$
$$ +\frac{(1-\rho_1)\ldots(1-\rho_p)}{(1-\alpha_1)\ldots(1-\alpha_p)}\,{}_{p+1}F_p\left[\begin{matrix}1,2-\rho_1,2-\rho_2,\ldots,2-\rho_p;\\ 2-\alpha_1,2-\alpha_2,\ldots,2-\alpha_p\end{matrix}\right], \quad (1.4) $$

and both the series on the right converge if

$$ R(\sum\rho-\sum\alpha-1)>0. $$

Similarly, when the argument is -1, the series ${}_pH_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p; & -1\\ \rho_1,\rho_2,\ldots,\rho_p & \end{matrix}\right]$ is convergent when $R(\sum\rho-\sum\alpha)>0$.

When $\rho_p=1$, the series ${}_pH_p$ evidently reduces to a series of the type ${}_pF_{p-1}$.

2. The connexion between Dougall's formulae and more general results

As we have seen, the formula (1.3) can be written in the form

$$ {}_3F_2\left[\begin{matrix}a,b,1;\\ c,d\end{matrix}\right]+\frac{(1-c)(1-d)}{(1-a)(1-b)}\,{}_3F_2\left[\begin{matrix}2-c,2-d,\ 1;\\ 2-a,2-b\end{matrix}\right] $$
$$ =\frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)}, \quad (2.1) $$

and this is the particular case when $c=1$ of the formula*

$$ {}_3F_2\left[\begin{matrix}a,b,c;\\ e,f\end{matrix}\right]=\frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(c-b)}{\Gamma(e-b)\Gamma(f-b)\Gamma(1+b-a)\Gamma(c)}\,{}_3F_2\left[\begin{matrix}b,b-e+1,b-f+1;\\ 1+b-c,1+b-a\end{matrix}\right] $$
$$ +\text{ a similar expression with } b \text{ and } c \text{ interchanged.} \quad (2.2) $$

Another formula given by Dougall in the same paper† can be written in the form

$$ {}_5H_5\left[\begin{matrix}1+\tfrac{1}{2}a, & b, & c, & d, & e;\\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\right] $$
$$ =\frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1-e)\Gamma(1+2a-b-c-d-e)}{\Gamma(1+a)\Gamma(1-a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)}\times $$
$$ \times\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a-b-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)\Gamma(1+a-d-e)}, \quad (2.3) $$

* See, for instance, Bailey (**1**), § 3.2 (2). Cf. Hardy (**3**), § 8.

† Dougall (**2**), § 14.

which sums a well-poised ${}_5H_5$ with the usual special parameter, and reduces when $e = a$ to

$$
{}_5F_4\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\right]
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-b-d)\Gamma(1+a-b-c)}. \tag{2.4}
$$

If d, e are replaced by $1+a-d$, $1+a-e$ in (2.3), and a is then made to tend to infinity, we obtain (1.3). Again, when $e = \frac{1}{2}a$, (2.3) reduces to

$$
{}_3H_3\left[\begin{matrix} b, & c, & d; \\ 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\right]
$$

$$
= \frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a-c-d)\Gamma(1+a-b-d)\Gamma(1+a-b-c)} \times
$$

$$
\times \frac{\Gamma(1-\frac{1}{2}a)\Gamma(1+\frac{1}{2}a)\Gamma(1+\frac{3}{2}a-b-c-d)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+\frac{1}{2}a-d)\Gamma(1+a)\Gamma(1-a)}, \tag{2.5}
$$

which sums a well-poised ${}_3H_3$, and is obviously an extension of Dixon's theorem which is given by putting $d = a$.

Until quite recently (2.3) has been something of a mystery to me, as it did not appear to be directly derivable from known transformations. A paper by Whipple has, however, brought the formula into relation with more general results. Whipple* gives numerous relations connecting three well-poised series of the type ${}_7F_6$. One of these relations is

$$
\frac{\psi[a;b,c,d,e,f]\sin\pi(b-f)}{\Gamma(b+f-a)\Gamma(1-c)\Gamma(1-d)\Gamma(1-e)}
$$

$$
= \frac{\psi[2f-a;f+b-a,f+c-a,f+d-a,f+e-a,f]\sin\pi(b-a)}{\Gamma(b)\Gamma(1+a-c-f)\Gamma(1+a-d-f)\Gamma(1+a-e-f)} -
$$

$$
- \frac{\psi[2b-a;b+f-a,b+c-a,b+d-a,b+e-a,b]\sin\pi(f-a)}{\Gamma(f)\Gamma(1+a-c-b)\Gamma(1+a-d-b)\Gamma(1+a-e-b)}, \tag{2.6}
$$

* Whipple (**4**). I am indebted to Dr. Whipple for his kindness in showing me the manuscript of this paper before publication.

where

$$\psi[a;b,c,d,e,f]$$

$$= \frac{\Gamma(1+a)}{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1-s)} \times$$

$$\times {}_7F_6\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d, & e, & f; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\right],$$

and $s \equiv b+c+d+e+f-2a-1$.

Writing $f = 1$ in (2.6), the second series ψ on the right reduces to a well-poised ${}_5F_4$ which can be summed by (2.4), and the formula obtained is equivalent to (2.3).

3. Another method

I now give another method of obtaining (2.3) which yields further results of a similar kind. The method simply consists in starting with a known transformation of a terminating series $\sum_{r=0}^{2n} u_r$, writing this as $\sum_{r=-n}^{n} u_{r+n}$, and, after suitable modifications, making n tend to infinity. Thus from a transformation of a terminating series we obtain a transformation of a series which is infinite in both directions.

We begin with Whipple's original formula* transforming a terminating well-poised ${}_7F_6$ into a Saalschützian ${}_4F_3$, namely,

$${}_7F_6\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d, & e, & -m; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m \end{matrix}\right]$$

$$= \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\, {}_4F_3\left[\begin{matrix} 1+a-b-c, & d, & e, & -m; \\ 1+a-b, & 1+a-c, & d+e-a-m \end{matrix}\right]. \tag{3.1}$$

Replace m, a, b, c, d, e by $2n, a-2n, b-n, c-n, d-n, e-n$, and the formula becomes

$$\sum_{r=-n}^{n} \frac{(a-2n)_{n+r}(1+\frac{1}{2}a-n)_{n+r}(b-n)_{n+r}(c-n)_{n+r}}{(n+r)!\,(\frac{1}{2}a-n)_{n+r}(1+a-b-n)_{n+r}(1+a-c-n)_{n+r}} \times$$

$$\times \frac{(d-n)_{n+r}(e-n)_{n+r}(-2n)_{n+r}}{(1+a-d-n)_{n+r}(1+a-e-n)_{n+r}(1+a)_{n+r}}$$

* See Bailey (1), § 4.3 (4).

$$=\frac{(-a)_{2n}(1+a-d-e)_{2n}}{(1+a-d-n)_{2n}(1+a-e-n)_{2n}}\times$$

$$\times\sum_{r=-n}^{n}\frac{(1+a-b-c)_{n+r}(d-n)_{n+r}(e-n)_{n+r}(-2n)_{n+r}}{(n+r)!(1+a-b-n)_{n+r}(1+a-c-n)_{n+r}(d+e-a-2n)_{n+r}},$$

which is equivalent to

$$\sum_{r=-n}^{n}\frac{(a-n)_r(1+\tfrac{1}{2}a)_r(b)_r(c)_r(d)_r(e)_r(-n)_r}{(n+1)_r(\tfrac{1}{2}a)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r(1+a-e)_r(1+a+n)_r}$$

$$=\frac{(1+a)_n(1-a)_n(1+a-d-e)_n(1+a-b-c)_n}{(1-b)_n(1-c)_n(1+a-d)_n(1+a-e)_n}\times$$

$$\times\sum_{r=-n}^{n}\frac{(1+a-b-c+n)_r(d)_r(e)_r(-n)_r}{(n+1)_r(1+a-b)_r(1+a-c)_r(d+e-a-n)_r}. \tag{3.2}$$

We shall now let $n\to\infty$ through positive integral values. The series of limits on the left of (3.2) is

$${}_5H_5\begin{bmatrix}1+\tfrac{1}{2}a, & b, & c, & d, & e; \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{bmatrix}, \tag{3.3}$$

which converges if $R(1+2a-b-c-d-e)>0$. To get the $(r+1)$th term of the original series from the corresponding term of (3.3) we have to multiply by $(a-n)_r(-n)_r/(n+1)_r(1+a+n)_r$.

The factor for the next term is obtained from this by multiplying by

$$\frac{(a-n+r)(-n+r)}{(n+1+r)(1+a+n+r)}=\frac{1-\{(1+a+2r)/(n+1+r)\}}{1+\{(1+a+2r)/(n-r)\}},$$

and this is less than 1 if $r>-\tfrac{1}{2}(1+a)$, assuming at present that a is real. Thus, if N is any positive integer greater than $-\tfrac{1}{2}(1+a)$, the value of the factor cannot exceed its value for $r=N$, and then it has a finite upper limit independent of n. We can thus apply Tannery's theorem to the part of the series on the left of (3.2) for which r is positive, and a similar argument can be used for the part of the series for which r is negative, and also for the series on the right of (3.2). We thus obtain the formula

$${}_5H_5\begin{bmatrix}1+\tfrac{1}{2}a, & b, & c, & d, & e; \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{bmatrix}$$

$$=\frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1-a)\Gamma(1+a-d-e)\Gamma(1+a-b-c)}\times$$

$$\times\,{}_2H_2\begin{bmatrix}d, & e; \\ 1+a-b, & 1+a-c\end{bmatrix}. \tag{3.4}$$

An appeal to the theory of analytic continuation shows that (3.4) is true if only

$$R(1+2a-b-c-d-e) > 0.$$

The series on the right of (3.4) can now be summed by (1.3) and we obtain (2.3). Similarly, in the formula*

$$\begin{aligned}
&{}_9F_8\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d, & e, \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, \end{matrix}\right. \\
&\qquad\qquad\qquad\left.\begin{matrix} f, & g, & -m; \\ 1+a-f, & 1+a-g, & 1+a+m \end{matrix}\right] \\
&\quad = \frac{(1+a)_m(1+k-e)_m(1+k-f)_m(1+k-g)_m}{(1+k)_m(1+a-e)_m(1+a-f)_m(1+a-g)_m}\times \\
&\qquad \times {}_9F_8\left[\begin{matrix} k, 1+\frac{1}{2}k, & k+b-a, & k+c-a, & k+d-a, \\ \frac{1}{2}k, & 1+a-b, & 1+a-c, & 1+a-d, \end{matrix}\right. \\
&\qquad\qquad\qquad\left.\begin{matrix} e, & f, & g, & -m; \\ 1+k-e, & 1+k-f, & 1+k-g, & 1+k+m \end{matrix}\right], \qquad (3.5)
\end{aligned}$$

where $k = 1+2a-b-c-d$, and the parameters are subject to the restriction

$$b+c+d+e+f+g-m = 2+3a,$$

we replace m, a, b, c, d, e, f, g by $2n$, $a-2n$, $b-n$, $c-n$, $d-n$, $e-n$, $f-n$, $g+n$. The same process as before then gives†

$$\begin{aligned}
&{}_6H_6\left[\begin{matrix} 1+\frac{1}{2}a, & b, & c, & d, & e, & f; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\ -1\right] \\
&\quad = \frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1-a)\Gamma(1+a-e-f)\Gamma(1+a-c-d)\Gamma(1+a-b-d)}\times \\
&\qquad \times \frac{\Gamma(1+2a-b-c-d)}{\Gamma(1+a-b-c)}\,{}_3H_3\left[\begin{matrix} 1+2a-b-c-d, & e, & f; \\ 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\right]. \qquad (3.6)
\end{aligned}$$

In this formula a well-poised ${}_6H_6\,(-1)$ is transformed into a general ${}_3H_3$.

When $b = a$, (3.6) reduces to the known transformation‡

$$\begin{aligned}
&{}_6F_5\left[\begin{matrix} a, 1+\frac{1}{2}a, & c, & d, & e, & f; \\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\ -1\right] \\
&\quad = \frac{\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1+a-e-f)}\,{}_3F_2\left[\begin{matrix} 1+a-c-d, & e, & f; \\ 1+a-c, & 1+a-d \end{matrix}\right]. \qquad (3.7)
\end{aligned}$$

* Bailey (1), § 4.3 (7). † See the note at the end of the paper.

‡ See Bailey (1), § 4.4 (2).

The series on the right of (3.6) can, in certain circumstances, be summed by (2.5), and so we can sum the series*

$$ {}_6H_6\left[\begin{matrix} 1+\tfrac{1}{2}a, \tfrac{1}{2}+a-z, & z+x, & z-x, \\ \tfrac{1}{2}a, \quad \tfrac{1}{2}+z, & 1+a-z-x, & 1+a-z+x, \end{matrix}\right. $$
$$ \left.\begin{matrix} z+y, & z-y; \\ 1+a-z-y, & 1+a-z+y \end{matrix}\; -1\right]. $$

It will be noticed that the methods of this paper have given no example in which a Saalschützian H occurs.

4. Basic series

A consideration of similar results for basic hypergeometric series is suggested by Jacobi's classical formula

$$ \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{n^2} = \prod_{n=1}^{\infty} [(1-q^{2n})(1-aq^{2n-1})(1-q^{2n-1}/a)], \qquad (4.1) $$

in which, on the left, n takes all integral values, both positive and negative, whereas the formulae connected with the Rogers-Ramanujan identities† only involve series for which $n \geqslant 0$.

By analogy with Whipple's work we write

$$ \chi[a;b,c,d,e,f] = \prod_{n=1}^{\infty}\left[\frac{(1-aq^n/b)(1-aq^n/c)(1-aq^n/d)\times \times(1-aq^n/e)(1-aq^n/f)(1-a^2q^{n+1}/bcdef)}{1-aq^n}\right]\times $$
$$ \times {}_8\Phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}\; a^2q^2/bcdef\right]. $$

Then it is known that‡

$$ \chi[a;b,c,d,e,f] = \prod_{n=1}^{\infty}\left[\frac{(1-aq^n/de)(1-aq^n/df)(1-aq^n/ef)\times \times(1-aq^n/b)(1-aq^n/c)(1-a^2q^{n+1}/bcdef)}{1-aq^n/def}\right]\times $$
$$ \times {}_4\Phi_3\left[\begin{matrix} aq/bc, & d, & e, & f; \\ aq/b, & aq/c, & def/a & \end{matrix}\; q\right]+ $$
$$ +\prod_{n=1}^{\infty}\left[\frac{(1-aq^n/bc)(1-dq^{n-1})(1-eq^{n-1})(1-fq^{n-1})\times \times(1-a^2q^{n+1}/bdef)(1-a^2q^{n+1}/cdef)}{1-q^{n-2}def/a}\right]\times $$
$$ \times {}_4\Phi_3\left[\begin{matrix} aq/ef, & aq/df, & aq/de, & a^2q^2/bcdef; \\ aq^2/def, & a^2q^2/bdef, & a^2q^2/cdef & \end{matrix}\; q\right] \qquad (4.2) $$

* Cf. Whipple (**5**), (14.1). † See, for instance, Bailey (**1**), § 8.6.
‡ Bailey (**1**), § 8.5 (3).

Now use this formula to transform

$$\chi[a^2q/def; b, c, aq/de, aq/df, aq/ef],$$

and we obtain exactly the same expression on the right as in (4.2). It follows that

$$\chi[a; b, c, d, e, f] = \chi[a^2q/def; b, c, aq/de, aq/df, aq/ef]. \tag{4.3}$$

Now repeat this formula, and we find that

$$\chi[a; b, c, d, e, f] = \chi[a^3q^2/b^2cdef; aq/bc, aq/bd, aq/be, aq/bf, a^2q^2/bcdef]. \tag{4.4}$$

The series on the right of (4.4) can be transformed by (4.2), and we obtain the formula

$$\chi[a; b, c, d, e, f] = \prod_{n=1}^{\infty}\left[\frac{(1-aq^n/cd)(1-aq^n/ce)(1-aq^n/cf)\times \times(1-aq^n/b)(1-bq^{n-1})(1-a^2q^{n+1}/bdef)}{1-bq^{n-1}/c}\right]\times$$

$$\times {}_4\Phi_3\left[\begin{matrix} aq/bd, & aq/be, & aq/bf, & c; \\ a^2q^2/bdef, & aq/b, & cq/b \end{matrix}\; q\right]+$$

$$+ \text{ a similar expression with } b \text{ and } c \text{ interchanged.} \tag{4.5}$$

The transformations given so far in this paragraph correspond to those given in my tract* for the ordinary type of well-poised series ${}_7F_6$. We now obtain three-term relations corresponding to some of those given by Whipple.

By means of (4.5) we can express

$$\chi[b^2/a; b, bc/a, bd/a, be/a, bf/a]$$

in terms of

$${}_4\Phi_3\left[\begin{matrix} q/c, & q/d, & q/e, & bf/a; \\ aq^2/cde, & bq/a, & fq/a \end{matrix}\; q\right] \quad \text{and} \quad {}_4\Phi_3\left[\begin{matrix} aq/cf, & aq/df, & aq/ef, & b; \\ a^2q^2/cdef, & bq/f, & aq/f \end{matrix}\; q\right].$$

Interchanging b and f in this relation, we express

$$\chi[f^2/a; f, fc/a, fd/a, fe/a, fb/a]$$

in terms of the first of the above ${}_4\Phi_3$'s and

$${}_4\Phi_3\left[\begin{matrix} aq/bc, & aq/bd, & aq/be, & f; \\ a^2q^2/bcde, & fq/b, & aq/b \end{matrix}\; q\right].$$

Again using (4.5) we can express

$$\chi[a; b, c, d, e, f]$$

* Bailey (1), § 7.5.

in terms of the last ${}_4\Phi_3$ and the second ${}_4\Phi_3$ in the first of these relations. From these three relations it is found that the three series ${}_4\Phi_3$ can all be eliminated, and we obtain the formula

$$\chi[a;b,c,d,e,f]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n/cf)(1-aq^n/df)(1-aq^n/ef)\times(1-bq^{n-1})(1-aq^n/b)(1-bq^{n-1}/a)}{(1-q^n/c)(1-q^n/d)(1-q^n/e)(1-bfq^{n-1}/a)\times(1-fq^n/b)(1-bq^{n-1}/f)}\right]\times$$

$$\times\chi[f^2/a;bf/a,cf/a,df/a,ef/a,f]+$$

$$+\text{ a similar expression with } b \text{ and } f \text{ interchanged.} \quad (4.6)$$

If we take $f = q$ in (4.6) the second series on the right reduces to a multiple of a well-poised ${}_6\Phi_5$ which can be summed by the basic analogue of (2.4), and the formula reduces to

$${}_6\Psi_6\left[\begin{matrix} q\surd a, & -q\surd a, & b, & c, & d, & e; \\ \surd a, & -\surd a, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix} a^2q/bcde\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/bc)(1-aq^n/bd)(1-aq^n/be)(1-aq^n/cd)}{(1-q^n/b)(1-q^n/c)(1-q^n/d)(1-q^n/e)(1-aq^n/b)}\times\right.$$

$$\left.\times\frac{(1-aq^n/ce)(1-aq^n/de)(1-q^n)(1-q^n/a)}{(1-aq^n/c)(1-aq^n/d)(1-aq^n/e)(1-a^2q^n/bcde)}\right], \quad (4.7)$$

where $${}_r\Psi_r\left[\begin{matrix}\alpha_1, \alpha_2,\dots, \alpha_r; \\ \rho_1, \rho_2,\dots, \rho_r\end{matrix} z\right] = \sum_{n=-\infty}^{\infty}\frac{(\alpha_1)_{q,n}(\alpha_2)_{q,n}\cdots(\alpha_r)_{q,n}}{(\rho_1)_{q,n}(\rho_2)_{q,n}\cdots(\rho_r)_{q,n}}z^n,$$

and $$(a)_{q,n} = (1-a)(1-aq)\dots(1-aq^{n-1}),$$

$$(a)_{q,-n} = 1/[(1-a/q)(1-a/q^2)\dots(1-a/q^n)].$$

This is the analogue of (2.3) for basic series. When $d = \surd a$, $e = -\surd a$, it reduces to

$${}_2\Psi_2\left[\begin{matrix} b, & c; \\ aq/b, & aq/c\end{matrix} -aq/bc\right] \quad (4.8)$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n/bc)(1-aq^{2n}/b^2)(1-aq^{2n}/c^2)\times(1-q^{2n})(1-aq^{2n-1})(1-q^{2n-1}/a)}{(1-q^n/b)(1-q^n/c)(1-aq^n/b)(1-aq^n/c)(1+aq^n/bc)}\right],$$

and when b and c both tend to infinity, this gives Jacobi's result (4.1).

5. Further remarks on basic series

The analogue to Whipple's fundamental three-term relation is

$$\prod_{n=1}^{\infty}[(1-aq^n/def)(1-defq^{n-1}/a)(1-bdq^{n-1}/a)(1-beq^{n-1}/a)\times$$
$$\times(1-q^n/c)(1-bfq^{n-1}/a)]\,\chi[a;b,c,d,e,f]$$
$$=\prod_{n=1}^{\infty}[(1-aq^n/b)(1-bq^{n-1}/a)(1-aq^n/ef)(1-aq^n/df)(1-aq^n/de)\times$$
$$\times(1-a^2q^{n+1}/bcdef)]\,\chi[ef/c;e,f,aq/bc,aq/cd,ef/a]+$$
$$+\prod_{n=1}^{\infty}[(1-aq^n/bc)(1-dq^{n-1})(1-eq^{n-1})(1-fq^{n-1})(1-a^2q^n/bdef)\times$$
$$\times(1-bdefq^{n-1}/a^2)](b/a)\,\chi[b^2/a;b,bc/a,bd/a,be/a,bf/a]. \quad (5.1)$$

This can be proved in exactly the same way as Whipple proves his formula, by using (4.2) three times to express the functions χ in terms of series ${}_4\Phi_3$. The relation (5.1) can now be used to write down a relation connecting

$$\chi[f^2/a;bf/a,cf/a,df/a,ef/a,f], \qquad \chi[b^2/a;bf/a,bc/a,bd/a,be/a,b]$$

and $$\chi[ef/c;e,f,aq/bc,aq/cd,ef/a].$$

The last function χ can be eliminated from this relation and (5.1), and we obtain (4.6), except that the coefficient of the first series on the right of (4.6) is given in terms of two infinite products instead of one. Comparing the two formulae we obtain the identity

$$\prod_{n=1}^{\infty}[(1-aq^n/b)(1-bq^{n-1}/a)(1-aq^n/ef)(1-efq^{n-1}/a)\times$$
$$\times(1-aq^n/df)(1-dfq^{n-1}/a)(1-aq^n/bde)(1-bdeq^{n-1}/a)]$$
$$=\prod_{n=1}^{\infty}[(1-aq^n/f)(1-fq^{n-1}/a)(1-aq^n/be)(1-beq^{n-1}/a)\times$$
$$\times(1-aq^n/bd)(1-bdq^{n-1}/a)(1-aq^n/def)(1-defq^{n-1}/a)]-$$
$$-\frac{b}{a}\prod_{n=1}^{\infty}[(1-dq^{n-1})(1-q^n/d)(1-eq^{n-1})(1-q^n/e)(1-bq^n/f)\times$$
$$\times(1-fq^{n-1}/b)(1-a^2q^n/bdef)(1-bdefq^{n-1}/a^2)]. \quad (5.2)$$

If we take $b=a\sqrt{q}$, $d=-1$, $e=-1$, $f=-a\sqrt{q}$, and then replace q by q^2, this becomes

$$\Big[\prod_{n=1}^{\infty}(1-q^{2n-1})\Big]^8+16q\Big[\prod_{n=1}^{\infty}(1+q^{2n})\Big]^8=\Big[\prod_{n=1}^{\infty}(1+q^{2n-1})\Big]^8, \quad (5.3)$$

another of Jacobi's classical results.

In the notation of theta functions (5.2) can be written

$$\vartheta_3(\alpha)\vartheta_3(\beta+\gamma)\vartheta_3(\beta+\delta)\vartheta_3(\alpha+\gamma+\delta)$$
$$=\vartheta_3(\beta)\vartheta_3(\alpha+\gamma)\vartheta_3(\alpha+\delta)\vartheta_3(\beta+\gamma+\delta)+$$
$$+\vartheta_1(\gamma)\vartheta_1(\delta)\vartheta_1(\beta-\alpha)\vartheta_1(\alpha+\beta+\gamma+\delta). \quad (5.4)$$

6. The alternative method for basic series

Results similar to those of § 3 are true for basic series. Thus, by an exactly similar process, we find from the analogue* of (3.5) that

$$\sum_{r=-\infty}^{\infty} \frac{(q\sqrt{a})_{q,r}(-q\sqrt{a})_{q,r}(b)_{q,r}(c)_{q,r}(d)_{q,r}(e)_{q,r}(f)_{q,r}(-1)^r}{(\sqrt{a})_{q,r}(-\sqrt{a})_{q,r}(aq/b)_{q,r}(aq/c)_{q,r}(aq/d)_{q,r}(aq/e)_{q,r}(aq/f)_{q,r}}\left(\frac{a^3}{bcdef}\right)^r q^{\frac{1}{2}r(r+3)}$$

$$= \Psi\begin{bmatrix} a^2q/bcd, & e, & f; & \\ aq/b, & aq/c, aq/d & & aq/ef \end{bmatrix} \times$$

$$\times \prod_{r=1}^{\infty}\left[\frac{(1-aq^r)(1-q^r/a)(1-aq^r/cd)(1-aq^r/bd)(1-aq^r/bc)(1-aq^r/ef)}{(1-aq^r/e)(1-aq^r/f)(1-q^r/b)(1-q^r/c)(1-q^r/d)(1-a^2q^r/bcd)}\right]. \tag{6.1}$$

From this formula a direct generalization of one of the Rogers-Ramanujan identities can be obtained, but it involves two products on the right instead of one, and is consequently not nearly so elegant.

It is worth while noting that the above method, when applied to Watson's transformation of basic series,† gives (4.7) from which Jacobi's formula (4.1) can be derived. Thus Jacobi's formula can be deduced from Watson's.

[*Added* 16 *December*, 1935. The formula (3.6) is true only if the series both terminate below, that is if one of the parameters $1+a-b$, $1+a-c$, $1+a-d$ is a positive integer. In the general case, when this restriction is not imposed, we obtain an extra series ${}_3F_2$ on the right of (3.6). This is due to the fact that when n is large (before proceeding to the limit) the terms near both ends of the series on the right are significant, while the terms in the middle are negligible. We therefore reverse the second half of the series before proceeding to the limit.‡

As a consequence of this restriction the last series ${}_6H_6$ occurring in § 3 can be summed only if the series terminates below, and the formula (6.1) is subject to the same restriction.]

* Bailey (**1**), § 8.5 (1). † See ibid., § 8.5 (2).

‡ Cf. Bailey (**1**), § 4.4 (4) where a similar phenomenon occurs.

REFERENCES

1. W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge Tract, 1935).
2. J. Dougall, *Proc. Edinburgh Math. Soc.* 25 (1907), 114–32.
3. G. H. Hardy, *Proc. Cambridge Phil. Soc.* 21 (1923), 492–503.
4. F. J. W. Whipple, *Proc. London Math. Soc.* (2) 40 (1936), 336–44.
5. F. J. W. Whipple, *Proc. London Math. Soc.* (2) 24 (1926), 247–63.

TANGENTIAL PROPERTIES OF A PLANE SET OF POINTS

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1. THE graph of an arbitrary function $f(x)$ of one variable may be considered as a very specialized set of points in two dimensions. From this point of view we may obtain the classic properties of the derivates of $f(x)$ as special cases of more general theorems on the structure of any plane set of points. These general theorems are not new: the first section of Theorem 1 was proved in part by W. H. and G. C. Young,* the second is due to Denjoy,† whilst the third and most difficult part establishes a result recently stated by Kolmogorov and Verčenko.‡

Theorem 2 is a generalization of a theorem of Lusin. When we combine these results and apply them to the plane set of points $\{x, f(x)\}$, we obtain theorems which contain the Denjoy-Young-Saks theorems on derivates and which go further in describing the possible dispositions not only of the extreme derivates but also of the intermediate derivates.

Notation. We refer to points in a plane by complex coordinates $z = x+iy$. Let the points z satisfying the inequalities

$$0 < |z-z_0| < \rho, \qquad \theta < \operatorname{am}(z-z_0) < \phi,$$

be called the *sector* $S_\theta^\phi(z_0, \rho)$, and those satisfying the second inequality only be called the *sector* $S_\theta^\phi(z_0)$. We shall, on occasion, write $S_n(z, p, q, r)$ for $S_{p\pi/r}^{q\pi/r}(z, n^{-1})$, where p, q, r, n are integers.

Let E be any plane set of points. Then

(i) the point z_0 is a *boundary point* of E, if

  (a) z_0 is a limit point of E (not necessarily in E),

  (b) there is a sector $S_\theta^\phi(z_0, \rho)$ which contains no points of E;

* W. H. and G. C. Young, *Proc. London Math. Soc.* (2) 16 (1918), 337–51. See also Smidov and Verčenko, *Comptes rendus*, 200 (1933), 616–17.

† A. Denjoy, *J. de Math.* (7), 1 (1915), 105–240; at 147.

‡ A. Kolmogorov and J. Verčenko, *Comptes rendus Ac. Sc. U.S.S.R.* 4 (1934), 361–4. In this paper the authors deduce Theorem 1 (iii) from a theorem (stated without proof) on the limits of an arbitrary function. The proof given below establishes the property of a set of points directly, and the theorem of Kolmogorov and Verčenko on limits may be derived from it.

(ii) the set E has a *derivate ray in direction* θ (or, briefly, has a derivate θ) if, for every positive η and ρ, the sector $S_{\theta-\eta}^{\theta+\eta}(z_0, \rho)$ contains points of E;

(iii) the set E has a *void sector* (α, β) at z_0, if α and β are derivates, but no θ satisfying $\alpha < \theta < \beta$ is a derivate. Under these circumstances, given any positive δ, there is a positive number $\rho = \rho(\delta)$ such that $S_{\alpha+\delta}^{\beta-\delta}(z_0, \rho)$ contains no points of E.

The terms *linearly measurable* and *length*, as applied to a plane set, are used in the sense defined by Carathéodory.*

A curve will be said to be *non-oscillatory* if it is monotone with respect to some oblique axes in the plane; all such curves are, of course, rectifiable.

2. THEOREM 1.

(i) *All boundary points of a plane set E form a linearly measurable set lying on a countable infinity of non-oscillatory curves*;†

(ii) *the boundary points at which there is a void sector of magnitude greater than* π *form a countable set*;

(iii) *the boundary points at which there is a void sector of magnitude less than* π *form a set of zero length.*

Let B be the set of boundary points of E. Let $B_{p,q,r,n}$ denote the set of points z in B for which $E\,.\,S_n(z, p, q, r) = 0$, and $B'_{p,q,r,n}$ the set of points of $B_{p,q,r,n}$ for which $E\,.\,S_n(z, p-1, p, r)$ and $E\,.\,S_n(z, q, q+1, r)$ both have z as a limit point. Then $B = \sum B_{p,q,r,n}$, where summation extends over all positive integral values of p, q, r, n for which $0 < q-p < 2r$. Consider one such set $e = B_{p,q,r,n}$. If w is a point of E or a limit point of E, then

$$e\,.\,S_n(w, p+r, q+r, r) = 0; \tag{2.1}$$

for, if z is any point of $S_n(w, p+r, q+r, r)$, then w is a point of $S_n(z, p, q, r)$, and so this sector contains at least one point of E. Hence z does not belong to e. We deduce two results.

(*a*) *The set B is linearly measurable.* If z is a limit point of e, not in e, then $S_n(z, p, q, r)$ contains a point w of E. The sector $S_n(w, p+r, q+r, r)$ contains z and therefore contains points of e, in contradiction of (2.1). Hence all limit points of e belong to e: that

* C. Carathéodory, *Göttinger Nachr.* (1914), 404–26.

† That is, they form a regular set in the sense defined by A. Besicovitch, *Math. Ann.* 98 (1927), 422–64.

is, e is a closed set. It follows at once that the set B, being the sum of a countable infinity of closed sets, is linearly measurable.

(*b*) *The points of B lie on a countable infinity of non-oscillatory curves.* If z belongs to $e = B_{p,q,r,n}$, then

$$e.S_n(z,p,q,r) = 0, \qquad e.S_n(z,p+r,q+r,r) = 0. \tag{2.2}$$

Let η be a positive number, and take axes in the plane in directions $(p\pi/r)+\eta$ and $(q\pi/r)-\eta$. By (2.2) the points of e inside any circle of radius $\frac{1}{2}n^{-1}$ lie on a curve monotone with respect to these axes: that is, non-oscillatory and rectifiable. The whole plane may be covered by a countable infinity of circles of radius $\frac{1}{2}n^{-1}$; hence all the points of $B_{p,q,r,n}$ lie on a countable infinity of such curves. Finally, since $B = \sum B_{p,q,r,n}$, the set B has the same property.

Let B_1 be the set of boundary points of E at which there is a void sector of magnitude greater than π. Then $B_1 = \sum B_{p,q,r,n}$, where the summation is taken over all integral values of p, q, r, n such that $r < q-p < 2r$. We have seen above that, if z belongs to $B_{p,q,r,n}$ then

$$B_{p,q,r,n}\, S_n(z,p,q,r) = 0, \qquad B_{p,q,r,n}\, S_n(z,p+r,q+r,r) = 0.$$

If $q-p > r$, the two sectors $S_n(z,p,q,r)$ and $S_n(z,p+r,q+r,r)$ together form a complete circle surrounding z. The points of $B_{p,q,r,n}$ are therefore isolated, and, being isolated, form a countable set. It follows that the set B_1 is also countable.

Let B_2 be the set of boundary points of E with a void sector of magnitude less than π. Then $B_2 = \sum B'_{p,q,r,n}$ where summation is taken over all integral values of p, q, r, n for which $2 < p-q < r-6$. If* $|B_2| > 0$, then we can find p, q, r, n such that $B'_{p,q,r,n} = e'$ is of positive exterior measure; we can also find a circle C of radius n^{-1} such that $|C.e'| > 0$. By the first part of the theorem the points of $C.e'$ lie on a curve γ monotone (in the strict sense) with respect to axes OU, OV in directions $(q-1)\pi/r$, $(p+1)\pi/r$. Consider the set e'' of points of $C.e'$ at which (i) the curve γ has a tangent; (ii) the projection of $C.e'$ on to OU (by lines parallel to OV) has outer density 1. By well-known theorems $|e''| = |C.e'| > 0$. We prove that the set e'' does not exist and so, by means of a contradiction, establish the third part of the theorem.

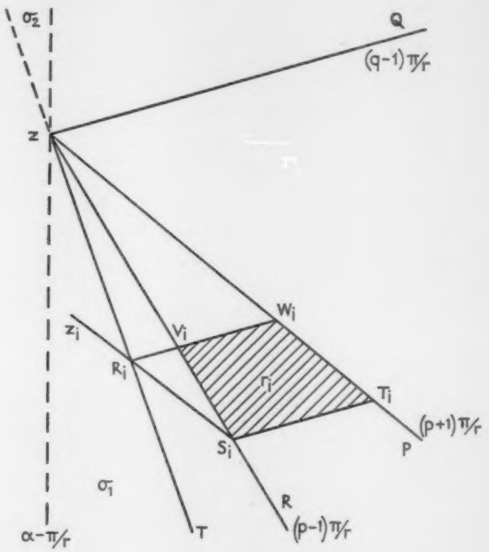
Let z be a point of e'', and let the tangent to γ at z be in direction α.

* If E is any set of points, $|E|$ denotes its external linear measure in the sense of Carathéodory.

All neighbouring points $z+re^{i\theta}$ of e' lie in one of the sectors σ_1, σ_2 given by

$$|\theta-\alpha| < \pi r^{-1}, \qquad |\theta-\alpha-\pi| < \pi r^{-1}.$$

Since $p-q < r-6$, at least one of the sectors σ_1, σ_2 is separated from $S_n(z, p-1, q+1, r)$ by an angle greater than or equal to πr^{-1}. Suppose this is true of σ_1, then $(p-1)\pi/r \geqslant \alpha+2\pi/r$. We can prove



that z is not a limit point of $E.S_n(z, p-1, p, r)$, contrary to the hypothesis that z belongs to $B'_{p,q,r,n}$.

Let zT, zQ, zP, zR be lines in directions $\alpha+\pi/r$, $(q-1)\pi/r$, $(p+1)\pi/r$, $(p-1)\pi/r$. Let z_i be a point of $\sigma_1 e''$: let $z_i R_i S_i$, parallel to zP, meet zT in R_i, zR in S_i; let $R_i V_i W_i$, parallel to zQ, meet zR in V_i, and zP in W_i: let $S_i T_i$ parallel to zQ meet zP in T_i. Since z_i belongs to e'', the portion of the quadrilateral $V_i W_i T_i S_i (= r_i)$ inside C is void of E. Now, if we can choose a sequence z_1, z_2, z_3,... tending to z, such that $zS_{i+1} > zV_i$, then the regions r_1, r_2, r_3,... will cover the whole of $S_n(z, p-1, p, r)$ in the neighbourhood of z. Since r_i is void of E, then z will not be a limit point of $E.S_n(z, p-1, p, r)$. The ratio

$$zS_i : zV_i : zR_i = \lambda : \mu : 1,$$

say, is independent of i. Hence we have the required condition, $zS_{i+1} > zV_i$, if

$$zR_{i+1} > (\mu/\lambda)zR_i. \tag{2.3}$$

The projection of the set $\sigma_1 e''$ on to the line zT has a point of outer density at z: therefore it is possible to choose z_1, z_2,... tending to z, in accordance with inequality (2.3). The last part of the theorem is thus proved.

3. Derivates of an arbitrary function

If the plane set of points E consists of the points $\{x, f(x)\}$ where $f(x)$ is an arbitrary function of x, Theorem 1 can be interpreted in terms of the derivates of $f(x)$ at points of continuity of the function. At points of discontinuity, however, at least one of the derivates D^+f, and so on, is infinite; and such derivates may not be derivates of the set of points in the sense we have defined. It is convenient to distinguish these two types of derivate by the symbols D and $\mathscr{D}$. We write, in the ordinary notation,

$$D^+ = \overline{\lim_{h\to+0}} \frac{f(x+h)-f(x)}{h},$$

and

$$\mathscr{D}^+ = \lim_{\delta\to 0} \overline{\lim_{h\to+0}} \frac{f_\delta(x+h)-f(x)}{h},$$

where $f_\delta(x+h)$ is equal to $f(x+h)$ if $|f(x+h)-f(x)| < \delta$, and is undefined elsewhere. There are similar definitions for D_-, $\mathscr{D}_-$, and so on. $\mathscr{D}^+$ is defined except in the countable set of points at which

$$\underline{\lim_{h\to+0}} |f(x+h)-f(x)| > 0.$$

D^+ and $\mathscr{D}^+$ differ only at points for which

$$\overline{\lim_{h\to+0}} f(x+h) > f(x):$$

at such points $D^+ = +\infty$, and $\mathscr{D}^+$ may have any value. If

$$\mathscr{D}^+ < D^+ = +\infty,$$

then, except in a countable set, $D_- = -\infty$, for, except in a countable set,

$$\overline{\lim_{h\to+0}} f(x+h) = \overline{\lim_{h\to+0}} f(x-h).$$

A line through $\{x, f(x)\}$ will be called *a derivate line of the function*, if it is a derivate line of the set E in the sense defined in § 1.

We have the following corollary of Theorem 1.

COROLLARY A. *If $f(x)$ is an arbitrary function of one variable, then the D-derivates are defined everywhere, and the $\mathscr{D}$-derivates are defined except in a countable set.*

I. *If D^+ is finite in a set G_1, then $D^+ = \mathscr{D}^+$ and*

(i) *the sub-set of G_1 in which $D^+ < D_-$ is countable;*

(ii) *at almost all points of G_1, $D^+ = D_- = \mathscr{D}_-$ and either*

(*a*) *there is a tangent (i.e. $D^+ = D_+ = D^- = D_-$),*

or (*b*) $\mathscr{D}^+ = \mathscr{D}_+ = \mathscr{D}^- = \mathscr{D}_-$ *and* $D^- = -D_+ = +\infty$, (3.1)

or (*c*) $\mathscr{D}^- = -\mathscr{D}_+ = +\infty$. (3.2)

II. *If $D^+ = +\infty$ and $\mathscr{D}^+$ is finite in a set G_2, then*

(i) *$D_- = -\infty$ and $\mathscr{D}^+ \geqslant \mathscr{D}_-$ in G_2 except for a countable set;*

(ii) *at almost all points of G_2, $\mathscr{D}_- = \mathscr{D}^+$ and either* (3.1) *or* (3.2) *above is true.*

At almost all points all lines between $\mathscr{D}^+$ and $\mathscr{D}_+$ and between $\mathscr{D}_-$ and $\mathscr{D}^-$ are derivate lines of the function.

To establish I (i) we observe that if $D^+ < D_-$ then $\mathscr{D}^+ < \mathscr{D}_-$, and by Theorem 1 (ii) this can be true only in a countable set. If $\mathscr{D}^+$ is finite we see by Theorem 1 (iii) that almost everywhere $\mathscr{D}^+ = \mathscr{D}_-$, and either $\mathscr{D}^- = -\mathscr{D}_+ = +\infty$ or $\mathscr{D}^+ = \mathscr{D}_+ = \mathscr{D}^- = \mathscr{D}_-$. If the former is true we have the case of I (ii) (*c*); if the latter, then either $D_+ = \mathscr{D}_+$ or $D_+ = -\infty$, and similarly either $D^- = \mathscr{D}^-$ or $D^- = -\infty$. But, as we have seen, if $\mathscr{D}_+$ is finite and $D_+ = -\infty$ then $D^- = +\infty$ except in a countable set, and conversely. Hence, except in this set, either I (ii) (*a*) or I (ii) (*b*) is true. The results under II are easily established in a similar manner.

If any one derivate is finite, this corollary gives the disposition of the derivates. The case in which all the derivates are infinite is included in the following theorem, which is a generalization of a theorem of Lusin.*

* See S. Saks, *Fund. Math.* 6 (1924), 111–16. The proof of Theorem 2 is an adaptation of that of Saks.

4. THEOREM 2.* *Let E be a given set of points and α a given number. Let E_α be the set of points at which the set E has a derivate α and a void sector $(\alpha-\eta, \alpha)$. Then the projection of E_α on to a line in direction $\alpha+\frac{1}{2}\pi$ is of measure zero.*

Suppose the projection of E_α is of positive measure for some value of α: without loss of generality we can take this value to be 0. Let OU, OV be axes in directions 0 and $\frac{1}{2}\pi$, and, if M is any plane set, let its projection on OV be denoted by M'. We have, by hypothesis, $|E_0'| > 0$.

Let e be the set of points of E_0 at which there are derivates 0 and $-\pi$, and $(-\pi, 0)$ is a void sector of E. Let e_n be the sub-set of points z of e at which $E.S_n(z, -3, -1, 4) = 0$. By Theorem 1, $|e'| = |E_\alpha'|$: and, since $e_n \prec e_{n+1}$ and $\lim e_n = e$, therefore

$$\lim_{n\to\infty} |e_n'| = |e'| > 0.$$

Hence we can take n so large that $|e_n'| > 0$. At all points z of e_n we have $e_n.S_n(z, 1, 3, 4) = 0$, as well as $e_n.S(z, -3, -1, 4) = 0$.

Let the whole plane be divided into squares of sides n^{-1} parallel to OU, OV: let the set of e_n in the rth square be $e_{r,n}$. Since $e_n = \sum e_{r,n}$ and $e_n' = \sum e_{r,n}'$,

$$\sum |e_{r,n}'| \geqslant |e_n'| > 0.$$

Therefore we can find r such that $|e_{r,n}'| > 0$. We write $\mathcal{E} = e_{r,n}$, and we prove that $|\mathcal{E}'| = 0$, so obtaining a contradiction, and establishing the theorem.

If $w = u+iv$ is a point of $\mathcal{E}$, then, given η, we can, by the definition of e, find ρ such that

$$E.S_{-\pi+\eta}^{-\eta}(w, \rho) = 0.$$

Given any positive number ϵ, we define $\mathcal{E}_m$ as the set of points w of $\mathcal{E}$ such that, if $w'(= u'+iv')$ belongs to E and $0 < u'-u < m^{-1}$, then $v'-v > -\epsilon(u'-u)$. It is easily seen that $\mathcal{E}_m \to \mathcal{E}$, and so $|\mathcal{E}_m'| \to |\mathcal{E}'|$: hence we may choose m so large that $|\mathcal{E}_m'| > \frac{1}{2}|\mathcal{E}'|$. At all points of $\mathcal{E}_m$ the set E has a derivate 0; consequently with any point w of $\mathcal{E}_m$ we can associate a point w' of E, arbitrarily near to w, such that

$$0 < u'-u < \tfrac{1}{2}m^{-1}, \qquad |v'-v| < \epsilon(u'-u). \tag{4.1}$$

The point w of $\mathcal{E}_m$ projects into the point v of $\mathcal{E}_m'$: with v we associate

* I am indebted to a referee for the remark that the proof of this theorem can be much simplified by use of Theorem 1. By this theorem it is sufficient to prove the result when E_α lies on a rectifiable curve: the proof is then a direct application of Lusin's theorem.

an interval $\{v-\epsilon(u'-u),\ v+\epsilon(u'-u)\}$ on OV. Since $(u'-u)$ can be taken arbitrarily small, such intervals cover the set $\mathcal{E}'_m$ in the sense of Vitali's theorem. We can, therefore, choose a finite non-overlapping set I of intervals (V_r, V'_r) $(r = 1, 2,..., p)$ such that

$$|\mathcal{E}_m . I| > \tfrac{1}{2}|\mathcal{E}_m| > \tfrac{1}{4}|\mathcal{E}|.$$

Suppose that the intervals of I are arranged in decreasing order of length: let Δ_r denote the interval (V_r, V'_r) and δ_r the corresponding u-interval $(u_r, u'_r) = \{u_r, u_r+(V'_r-V_r)/2\epsilon\}$. Then, if $r < s$, we have $|\Delta_r| > |\Delta_s|$, $|\delta_r| > |\delta_s|$.

Let Δ_r and Δ_s be two intervals such that δ_r and δ_s overlap, and suppose that $V_s > V'_r$. It is easily seen from (4.1) that the point $u_s+iv_s = u_s+\frac{1}{2}i(V_s+V'_s)$ satisfies the inequalities

$$0 < v_s - V'_r < \epsilon(u'_r - u_s) < 2\epsilon \max(\delta_r, \delta_s).$$

Hence $$V'_s - V'_r < 3\epsilon \max(\delta_r, \delta_s) = \tfrac{3}{2}\max(\Delta_r, \Delta_s). \tag{4.2}$$

If we omit from the sequence $\delta_1, \delta_2,..., \delta_p$ any interval which overlaps δ_1 and call the first remaining interval δ'_2, then by repeating the process we obtain a sequence $\delta'_1, \delta'_2,..., \delta'_q$, of non-overlapping intervals. Any of the omitted intervals of $\delta_1, \delta_2,..., \delta_p$ overlaps at least one of $\delta'_1, \delta'_2,..., \delta'_q$. If $\Delta'_1, \Delta'_2,..., \Delta'_q$ are the corresponding intervals on OV, then

$$\sum_1^q |\Delta'_i| = 2\epsilon \sum_1^q |\delta'_i| \leqslant \frac{2\epsilon}{n}.$$

Now, by (4.2), those intervals Δ_s $(s \geqslant r)$ for which δ_s overlaps δ_r cover a length less than $4|\Delta_r|$. Therefore

$$\sum_1^q |\Delta'_i| > \tfrac{1}{4}\sum_1^p |\Delta_r| > \tfrac{1}{16}|\mathcal{E}|.$$

Hence $|\mathcal{E}| \leqslant 32\epsilon n^{-1}$: since ϵ is arbitrary, $|\mathcal{E}| = 0$. This contradiction of our hypothesis establishes the theorem.

The following is an immediate corollary of Theorem 2.

COROLLARY B.*

If $\mathcal{D}^+ = D^+ = +\infty$, *then except in set of measure zero,* $\mathcal{D}_- = -\infty$, *and either*

(i) $\mathcal{D}^- = +\infty$, $\mathcal{D}_+ = -\infty$, *and derivate lines lie in every direction*; *or* (ii) $\mathcal{D}^-$ *and* $\mathcal{D}_+$ *are finite and equal, and the derivate lines fill a half-plane.*

Case (ii) of this corollary is, of course, included also in Corollary A.

* I am again much indebted to a referee who pointed out that my original statement of this corollary was palpably false.

THE POTENTIAL OF A SPHERE INSIDE AN INFINITE CIRCULAR CYLINDER

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1. Introduction

I PROPOSE to obtain a solution of the harmonic equation which shall have given values on a sphere, symmetrically placed inside an infinite circular cylinder on which its value shall be zero or constant. The method employed is to construct a series of functions satisfying the equation and having the required boundary conditions on the cylinder, and then to combine them to satisfy the conditions on the sphere. The conditions on the sphere will be supposed to be such as can be represented by means of Legendre polynomials, i.e. there is symmetry about the axis of the cylinder. In particular, I shall consider even functions, which implies the further condition that there is symmetry with respect to the plane through the centre of the sphere perpendicular to the axis of the cylinder. The case when the solution has a constant value on the sphere and zero value on the cylinder will be completely worked out and numerical values of the coefficients will be given for various values of the ratio of the radius of the sphere to the radius of the cylinder. I shall show that the solution converges if this ratio is not too great.

2. The potential functions

We require solutions of

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial \varpi^2}+\frac{1}{\varpi}\frac{\partial V}{\partial \varpi}+\frac{\partial^2 V}{\partial z^2} = 0, \tag{1}$$

where ϖ, z, θ are cylindrical coordinates with origin at the centre of the sphere $r = b$, the axis of z lying along the axis of the cylinder $\varpi = a$; and r is the radius vector, so that we have the relations

$$r\cos\theta = z, \qquad r\sin\theta = \varpi. \tag{2}$$

The boundary conditions to be satisfied are

(i) $V = 0$, when $\varpi = a$;

if V is to have some non-zero constant value, this constant can be added to the potential functions before satisfying the conditions on the sphere.

(ii) $V = \sum_{n=0}^{\infty} C_n P_n(\cos\theta)$, when $r = b$,

where the C_n are given constants and $P_n(\cos\theta)$ is the Legendre polynomial of order n. We consider only even functions so that C_n is zero when n is odd. The odd functions can be constructed by a similar method, if required.

Solutions of (1) are of the type

$$V = e^{\pm imz} I_0(m\varpi),$$

where $I_0(z)$ is the Bessel Function of zero order and imaginary argument. Since we require solutions even in z, we take

$$V = \cos mz\, I_0(m\varpi). \tag{3}$$

A more general solution, provided the integral converges, is

$$V = \int_0^\infty I_0(m\varpi)\cos mz f(m)\, dm, \tag{4}$$

where $f(m)$ is to be chosen so that the second boundary condition is satisfied, when the solution contains a singularity at the origin.

To obtain this singularity consider the function $V = 1/r$. This will produce values of V on $\varpi = a$ which are functions of z only, and these have to be cancelled by the integral (4). We therefore find $f(m)$ which shall produce on $\varpi = a$ the same values as $1/r$.

When $\varpi = a$, $1/r$ has the value

$$\phi(z) = (z^2+a^2)^{-\frac{1}{2}},$$

hence

$$\phi(z) = \int_0^\infty I_0(ma)\cos mz f(m)\, dm. \tag{5}$$

This is an integral equation to determine $f(m)$ and is solved by means of the reciprocal integral equations

$$\phi(z) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty F(m)\cos mz\, dm, \qquad F(m) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \phi(z)\cos mz\, dz.$$

Thus

$$f(m)I_0(ma) = \sqrt{\left(\frac{2}{\pi}\right)} F(m) = \frac{2}{\pi} \int_0^\infty \phi(z)\cos mz\, dz,$$

and so

$$f(m) = \frac{2}{\pi} \int_0^\infty \frac{\phi(z)\cos mz}{I_0(ma)}\, dz.$$

Hence $$V = \frac{2}{\pi}\int_0^\infty \frac{I_0(m\varpi)}{I_0(ma)}\cos mz\,dm \int_0^\infty \phi(z)\cos mz\,dz$$

$$= \frac{2}{\pi}\int_0^\infty \frac{I_0(m\varpi)}{I_0(ma)}\cos mz\,dm \int_0^\infty \frac{\cos mz}{\sqrt{(z^2+a^2)}}\,dz.$$

But $$\int_0^\infty \frac{\cos mz}{\sqrt{(z^2+a^2)}}\,dz = K_0(ma),$$

where $K_0(z)$ is the Bessel Function of second kind of zero order and imaginary argument.*

Thus $$V = \frac{2}{\pi}\int_0^\infty \frac{I_0(m\varpi)K_0(ma)\cos mz}{I_0(ma)}\,dm. \tag{6}$$

Now it may be shown that†

$$e^{r\cos\theta} J_0(r\sin\theta) = \sum_{n=0}^\infty \frac{r^n}{n!} P_n(\cos\theta).$$

In this equation replace $r\cos\theta$ by imz and $r\sin\theta$ by $im\varpi$ and obtain

$$e^{imz} I_0(m\varpi) = \sum_{n=0}^\infty \frac{(irm)^n}{n!} P_n(\cos\theta).$$

Equating real parts we get the relation

$$\cos mz\, I_0(m\varpi) = \sum_{n=0}^\infty (-)^n \frac{m^{2n}}{(2n)!} r^{2n} P_{2n}(\cos\theta). \tag{7}$$

Substitution of (7) in (6) gives

$$V = \frac{2}{\pi}\int_0^\infty \sum_{n=0}^\infty (-)^n \frac{m^{2n}K_0(ma)r^{2n}P_{2n}(\cos\theta)}{(2n)!\,I_0(ma)}\,dm.$$

Assuming that we may change the order of summation and integration, V becomes

$$V = \frac{2}{\pi}\sum_{n=0}^\infty (-)^n \frac{I_{2n}\, r^{2n} P_{2n}(\cos\theta)}{(2n)!\,a^{2n+1}}, \tag{8}$$

where $$I_{2n} = \int_0^\infty \frac{K_0(m)m^{2n}}{I_0(m)}\,dm. \tag{9}$$

The potential function given by (8) produces the same values on

* Mehler, *Math. Ann.* 18 (1881), 182; Watson, *Bessel Functions* (Cambridge, 1922), § 6.23. † E. W. Hobson, *Proc. London Math. Soc.* (1) 25 (1893), 66.

$\varpi = a$ as $1/r$, and hence a potential function satisfying the first boundary condition and having this singularity at the origin is

$$V_0 = \frac{1}{\rho} - \sum_{n=0}^{\infty} {}_0\alpha_{2n}\, \rho^{2n} P_{2n}(\cos\theta), \tag{10}$$

where
$$\rho = r/a, \tag{11}$$

and
$$_0\alpha_{2n} = \frac{(-1)^n}{(2n)!} \frac{2}{\pi} I_{2n}. \tag{12}$$

Further functions having similar properties are obtained by differentiating (10) an even number of times with respect to z. Hence we define

$$\begin{aligned} V_{2s} &= \frac{a^{2s}}{(2s)!} \frac{\partial^{2s} V_0}{\partial z^{2s}} \\ &= \frac{P_{2s}(\cos\theta)}{\rho^{2s+1}} - \sum_{n=0}^{\infty} \binom{2n+2s}{2s} {}_0\alpha_{2n+2s}\, \rho^{2n} P_{2n}(\cos\theta), \\ &= \frac{P_{2s}(\cos\theta)}{\rho^{2s+1}} - \sum_{n=0}^{\infty} {}_{2s}\alpha_{2n}\, \rho^{2n} P_{2n}(\cos\theta), \end{aligned} \tag{13}$$

where
$$_{2s}\alpha_{2n} = \binom{2n+2s}{2s} {}_0\alpha_{2n+2s} = \frac{(-1)^{n+s}}{(2n)!\,(2s)!} \frac{2}{\pi} I_{2n+2s}. \tag{14}$$

3. The determination of the constants

We take as our general potential function

$$\begin{aligned} V = \sum_{s=0}^{\infty} A_{2s} V_{2s} &= \sum_{s=0}^{\infty} A_{2s} \left\{ \frac{P_{2s}(\cos\theta)}{\rho^{2s+1}} - \sum_{n=0}^{\infty} {}_{2s}\alpha_{2n}\, \rho^{2n} P_{2n}(\cos\theta) \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \frac{A_{2n}}{\rho^{2n+1}} - \rho^{2n} \sum_{s=0}^{\infty} {}_{2s}\alpha_{2n}\, A_{2s} \right\} P_{2n}(\cos\theta). \end{aligned} \tag{15}$$

Let $\lambda = b/a$; then, when $\rho = \lambda$, V must satisfy

$$V = \sum_{n=0}^{\infty} C_{2n} P_{2n}(\cos\theta). \tag{16}$$

Equating the coefficients of $P_{2n}(\cos\theta)$ in (15) and (16), we have the following equations between the constants A_{2n},

$$A_{2n} \lambda^{-2n-1} - \lambda^{2n} \sum_{s=0}^{\infty} {}_{2s}\alpha_{2n} A_{2s} = C_{2n} \quad (n \geqslant 0). \tag{17}$$

These equations are solved by a method of successive approximation.

Let $$A_{2n} = \sum_{r=0}^{\infty} A_{2n}^{(r)} \quad (n \geqslant 0) \tag{18}$$

where $$A_{2n}^{(0)} = \lambda^{2n+1} C_{2n}, \tag{19.1}$$

$$A_{2n}^{(r)} = \lambda^{4n+1} \sum_{s=0}^{\infty} {}_{2s}\alpha_{2n} A_{2s}^{(r-1)} \quad (r \geqslant 1). \tag{19.2}$$

The convergence of this process will be discussed later.

4. The evaluation of the integrals

We now consider the integrals defined by equation (9),

$$I_{2n} = \int_0^{\infty} \frac{K_0(m) m^{2n}}{I_0(m)}\, dm. \tag{9}$$

The functions $K_0(m)$, $I_0(m)$ have the well-known expansions

$$K_0(m) = -I_0(m) \log \tfrac{1}{2}m + \sum_{t=0}^{\infty} \frac{(\frac{1}{2}m)^{2t}}{(t!)^2} \psi(t+1), \qquad I_0(m) = \sum_{t=0}^{\infty} \frac{(\frac{1}{2}m)^{2t}}{(t!)^2},$$

where $$\psi(t+1) = \sum_{p=1}^{t} \frac{1}{p} - \gamma \quad (t > 0), \qquad \psi(1) = -\gamma,$$

and γ is Euler's constant.

Hence near the origin the integrand behaves like $m^{2n} \log m$ and can be integrated. The function $I_0(m)$ is an increasing function while $K_0(m)$ decreases, and for large arguments we have the asymptotic expressions

$$K_0(m) \sim \sqrt{\left(\frac{\pi}{2m}\right)} e^{-m}, \qquad I_0(m) \sim \sqrt{\left(\frac{1}{2\pi m}\right)} e^{m},$$

so that the integral converges at the upper limit. The series given above cannot be used for the evaluation of the integral except in the first part of the range, while the asymptotic series are of no use below $m = 5$. We therefore divide the range of integration into three parts (i) $0 < m \leqslant 1$, (ii) $1 \leqslant m \leqslant 5$, (iii) $5 \leqslant m < \infty$.

(i) *The interval* $0 < m \leqslant 1$.

Within this range the series given above are used to obtain an expansion of $K_0(m)/I_0(m)$. Both series are convergent within the range and may be divided, leading to the expansion

$$\frac{K_0(m)}{I_0(m)} = -\log x - \gamma + \sum_{r=1}^{\infty} a_r x^{2r} \quad (x = \tfrac{1}{2}m).$$

The coefficients a_r are given in Table I and are correct to eight figures. The value obtained for $K_0(m)/I_0(m)$, by using them in the

worst case i.e. $x = \frac{1}{2}$, is in agreement to eight figures with that obtained by division of the tabulated values of the functions.

TABLE I. *Coefficients in the expansions*

r	a_r	b_r
0	$\gamma = 0{\cdot}57721566$	1
1	1·0	−0·25
2	−0·625	0·03125
3	0·42592593	−0·1328125
4	−0·29383681	0·03271484
5	0·20313889	−0·42321777
6	−0·14049061	0·11360168
7	0·09717012	
8	−0·06720852	
9	0·04648543	
10	−0·03215213	

Hence

$$I'_{2n} = \int_0^1 \frac{K_0(m)m^{2n}}{I_0(m)}\,dm$$

$$= \int_0^1 \Big\{-\log\tfrac{1}{2}m-\gamma+\sum_{r=1}^{\infty} a_r(\tfrac{1}{2})^{2r}m^{2r}\Big\}m^{2n}\,dm$$

$$= \frac{1}{(2n+1)^2}+\frac{1}{(2n+1)}(\log 2-\gamma)+\sum_{r=1}^{\infty} a_r(\tfrac{1}{2})^{2r}\frac{1}{2r+2n+1}. \quad (20)$$

The values of I'_{2n} are given in the first column of Table II.

(ii) *The interval* $1 \leqslant m \leqslant 5$.

For computation in the second range a method of numerical integration is used. The values of the integrands were tabulated at intervals of 0·1 using the tables of $K_0(m)$, $I_0(m)$ given by Aldis.* Tables of differences were constructed and Gregory's† formula used in each quarter of the range. Differences up to the seventh were used which gave the values of I''_{2n} correct to six figures in most cases, five in the remainder. Some of the integrals were checked by Weddle's Rule,‡ giving agreement to four figures. The values of I''_{2n} are given in the second column of Table II.

* W. S. Aldis, *Proc. Roy. Soc.* A 64 (1899), 203.

† Whittaker and Robinson, *Calculus of Observations* (Blackie 1924), 143.

‡ Whittaker and Robinson, ibid., 151.

(iii) *The interval* $5 \leqslant m < \infty$.

When m is large the asymptotic formulae for the functions may be used and these are

$$K_0(m) = \sqrt{\left(\frac{\pi}{2m}\right)}e^{-m}\left\{1-\frac{1}{8m}+\frac{9}{2!\,(8m)^2}-\frac{9.25}{3!\,(8m)^3}+\frac{9.25.49}{4!\,(8m)^4}-...\right\} \tag{21.1}$$

$$I_0(m) = \sqrt{\left(\frac{1}{2\pi m}\right)}e^{m}\left\{1+\frac{1}{8m}+\frac{9}{2!\,(8m)^2}+\frac{9.25}{3!\,(8m)^3}+\frac{9.25.49}{4!\,(8m)^4}+...\right\} \tag{21.2}$$

Although these series are not convergent, if we take each of them as far as the term in m^{-6} and then treat them as polynomials, the resulting error is negligible to our degree of accuracy. The series obtained by dividing these polynomials is

$$\frac{K_0(m)}{I_0(m)} = \pi e^{-2m}\sum_{r=0}^{\infty} b_r\, m^{-r}.$$

The values of the coefficients b_r are given in Table I. The value of $K_0(m)/I_0(m)$ obtained by using this series is correct to seven decimal places in the worst case i.e. $m = 5$. Then

$$I'''_{2n} = \int\limits_5^\infty \frac{K_0(m)}{I_0(m)}m^{2n}\,dm = \pi\int\limits_5^\infty e^{-2m}\sum_{r=0}^{\infty} b_r\, m^{2n-r}\,dm$$

$$= \pi\sum_{r=0}^{\infty} b_r\int\limits_5^\infty e^{-2m}m^{2n-r}\,dm.$$

The values of I'''_{2n} and of I_{2n} itself are tabulated in Table II. These are correct to five figures, or six where given.

TABLE II. *Values of the integrals*

n	I'_{2n}	I''_{2n}	I'''_{2n}	I_{2n}
0	1·192291	0·175318	$6{\cdot}8\times10^{-5}$	1·36768
1	0·194823	0·449990	$2{\cdot}0774\times10^{-3}$	0·64689
2	0·095089	1·908879	$6{\cdot}5894\times10^{-2}$	2·06986
3	0·061642	$1{\cdot}39054\times10^{1}$	2·2014	$1{\cdot}6168\times10^{1}$
4	0·04533	$1{\cdot}52698\times10^{2}$	$7{\cdot}8951\times10^{1}$	$2{\cdot}3169\times10^{2}$
5		$2{\cdot}16787\times10^{3}$	$3{\cdot}1187\times10^{3}$	$5{\cdot}2866\times10^{3}$
6		$3{\cdot}59463\times10^{4}$	$1{\cdot}40102\times10^{5}$	$1{\cdot}7605\times10^{5}$

The values of the coefficients ${}_{2p}\alpha_{2n}$ are now determined using equations (12) and (14). They are given in Table III.

TABLE III. *Values of* $(-)^{n+p}{}_{2p}\alpha_{2n}$

p	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
0	$8{\cdot}7069\times 10^{-1}$	$2{\cdot}0591\times 10^{-1}$	$5{\cdot}4905\times 10^{-2}$	$1{\cdot}4296\times 10^{-2}$	$3{\cdot}6582\times 10^{-3}$	$9{\cdot}2749\times 10^{-4}$	$2{\cdot}3398\times 10^{-4}$
1	$2{\cdot}0591\times 10^{-1}$	$3{\cdot}2943\times 10^{-1}$	$2{\cdot}1444\times 10^{-1}$	$1{\cdot}0243\times 10^{-1}$	$4{\cdot}1736\times 10^{-2}$	$1{\cdot}5442\times 10^{-2}$	
2	$5{\cdot}4905\times 10^{-2}$	$2{\cdot}1444\times 10^{-1}$	$2{\cdot}5608\times 10^{-1}$	$1{\cdot}9477\times 10^{-1}$	$1{\cdot}1582\times 10^{-1}$		
3	$1{\cdot}4296\times 10^{-2}$	$1{\cdot}0243\times 10^{-1}$	$1{\cdot}9477\times 10^{-1}$	$2{\cdot}1620\times 10^{-1}$			
4	$3{\cdot}6582\times 10^{-3}$	$4{\cdot}1736\times 10^{-2}$	$1{\cdot}1582\times 10^{-1}$				
5	$9{\cdot}2749\times 10^{-4}$	$1{\cdot}5442\times 10^{-2}$					
6	$2{\cdot}3398\times 10^{-4}$						

5. Inequalities for the coefficients

We first require an inequality for the integral I_{2n}. This is not easily obtained as we have no expansion that is valid throughout the whole range of integration. To overcome this difficulty we divide the range into two parts, (0, 1) and (1, ∞). In the part (0, 1) the value of the integral I'_{2n} is given by equation (20), from which we deduce

$$|I'_{2n}| < \frac{1}{(2n+1)^2}+\frac{1}{(2n+1)}(\log 2-\gamma)+\frac{a_1}{2n+1}$$

$$< 2+\log 2-\gamma = \alpha. \tag{22}$$

In the range beyond $m = 5$ we may use the asymptotic expansions (21), which lead to the inequality

$$\frac{K_0(m)}{I_0(m)} < \pi e^{-2m}. \tag{23}$$

In the range (1, 5) we have no expansion, but $K_0(m)/I_0(m)$ steadily decreases, and it is found, from numerical values, that the inequality (23) holds good. Hence

$$|I''_{2n}|+|I'''_{2n}| < \pi\int\limits_1^\infty m^{2n}e^{-2n}\,dm$$

$$< \frac{(2n+1)!\,\pi e^{-2}}{2^{2n+1}}.$$

We have therefore the required inequality

$$|I_{2n}| < \alpha+\frac{(2n+1)!\,\pi e^{-2}}{2^{2n+1}}.$$

Returning to equation (14) for the coefficients,

$$\begin{aligned} |{}_{2s}\alpha_{2n}| &\leqslant \frac{2|I_{2n+2s}|}{\pi(2n)!\,(2s)!} \\ &< \frac{2\alpha}{\pi(2n)!\,(2s)!}+\frac{(2n+2s+1)!\,e^{-2}}{2^{2n+2s}(2n)!\,(2s)!} \\ &< \frac{2\alpha}{\pi}+(2s+1)e^{-2}, \end{aligned} \tag{24}$$

since

$$\binom{2n+2s+1}{2n} \leqslant 2^{2n+2s}.$$

6. The convergence of the solution

To investigate the convergence of the approximation process we assume that

$$|A_{2n}^{(r-1)}| < K_{r-1}\lambda^{4n+1} \tag{25}$$

where K_{r-1} is a constant. When $r = 0$ this is satisfied if $|C_{2n}| < K_0\lambda^{2n}$ for all n.

This restriction on the C's is not important in practical cases and the condition is satisfied in the example that has been worked out.

Equation (19.2) now leads to the inequality

$$\begin{aligned} |A_{2n}^{(r)}| &< \lambda^{4n+1}\sum_{s=0}^{\infty} |{}_{2s}\alpha_{2n}|\,.\,|A_{2s}^{(r-1)}| \\ &< \lambda^{4n+1}\sum_{s=0}^{\infty} K_{r-1}\lambda^{4s+1}\left(\frac{2\alpha}{\pi}+\{2s+1\}e^{-2}\right) \\ &= K_{r-1}\lambda^{4n+1}\left\{\frac{2\alpha\lambda}{\pi(1-\lambda^4)}+\frac{e^{-2}(1+\lambda^4)\lambda}{(1-\lambda^4)^2}\right\} \\ &= K_r\lambda^{4n+1}, \end{aligned}$$

where

$$K_r = K_{r-1}\left\{\frac{2\alpha\lambda}{\pi(1-\lambda^4)}+\frac{e^{-2}(1+\lambda^4)\lambda}{(1-\lambda^4)^2}\right\},$$

so that $\sum_{r=0}^{\infty} A_{2n}^{(r)}$ converges if

$$\frac{2\alpha\lambda}{\pi(1-\lambda^4)}+\frac{e^{-2}(1+\lambda^4)\lambda}{(1-\lambda^4)^2} < 1.$$

The best value of λ satisfying this inequality is $\lambda = 0{\cdot}58$, so that we have convergence if

$$\lambda < 0{\cdot}58.$$

Returning to equation (15) for V, we substitute for the constants A_{2n} from (18)

$$V = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{2s}^{(r)} V_{2s}.$$

The partial sums with respect to s will converge absolutely if

$$|A_{2n}^{(r)}|\rho^{-2n-1}+\rho^{2n}\sum |{}_{2s}\alpha_{2n}|\,.\,|A_{2s}^{(r)}|$$

converges. Since V_{2s} is convergent if $\rho < 1$, i.e. if the sphere lies inside the cylinder, the above series converges. The potential function itself will converge provided the $A_{2n}^{(r)}$ converge to A_{2n} and this has been proved for $\lambda < 0{\cdot}58$.

7. The values of the constants

I have calculated the numerical values of the constants in the particular case when the potential of the sphere is constant and equal to unity. Here $C_0 = 1, \quad C_{2n} = 0 \quad (n \geqslant 1).$

Equations (19) become

$$A_0^{(0)} = \lambda, \qquad A_{2n}^{(0)} = 0 \quad (n \geqslant 1),$$

$$A_{2n}^{(r)} = \lambda^{4n+1}\sum_{s=0}^{\infty} {}_{2s}\alpha_{2n} A_{2s}^{(r-1)} \quad (r \geqslant 1).$$

The values of A_{2n} are given in Table IV for different values of λ.

TABLE IV. *Values of* A_{2n}

λ	0·1	0·2	0·3	0·4	0·5
A_0	$1{\cdot}095\times10^{-1}$	$2{\cdot}422\times10^{-1}$	$4{\cdot}061\times10^{-1}$	$6{\cdot}139\times10^{-1}$	$8{\cdot}865\times10^{-1}$
$-A_2$	$2{\cdot}26\times10^{-7}$	$1{\cdot}60\times10^{-5}$	$2{\cdot}03\times10^{-4}$	$1{\cdot}30\times10^{-3}$	$5{\cdot}76\times10^{-3}$
A_4			$4{\cdot}40\times10^{-7}$	$8{\cdot}91\times10^{-6}$	$9{\cdot}75\times10^{-5}$
$-A_6$				$5{\cdot}98\times10^{-8}$	$1{\cdot}62\times10^{-6}$
A_8					$1{\cdot}95\times10^{-8}$

The final series is easily rearranged in the form

$$V = 1+\sum_{n=0}^{\infty} B_{2n}\left\{\left(\frac{\lambda}{\rho}\right)^{2n+1}-\left(\frac{\rho}{\lambda}\right)^{2n}\right\}P_{2n}(\cos\theta) \quad (B_{2n} = A_{2n}\lambda^{-2n-1}).$$

The capacity of the condenser formed by the sphere and the cylinder is readily seen to be κA_0 where κ is the specific inductive capacity of the medium between them.

I should like to thank Professor R. C. J. Howland for much helpful criticism and advice.

/ THE EULERIAN FUNCTIONS OF A GROUP

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THE present paper arose from an attempt to find a formula for the number of ways of generating the icosahedral group from a given number of its elements. This problem (and the similar problem which occurs when the orders of some or all of the generators are specified) may be solved (obviously) for any group whose sub-groups are sufficiently known. The most convenient way of doing this seems to be by an extension to arbitrary groups of finite order of the principle of enumeration for prime-power groups given in *Proc. London Math. Soc.* (2) 36 (1933), 39.

In § 1 we define the particular invariants of a group which we are to calculate (they form a natural generalization of the Eulerian function $\phi(n)$ of arithmetic), and point out their significance for the structure theory of the free groups and similar types of infinite group.

In § 2 we prove the generalized enumeration principle* already referred to (it includes as other particular cases the well-known inversion formulae† of the elementary theory of numbers); and give rules which simplify the calculation of the coefficients in certain cases, of which Theorem 2.3 is the most useful.

In § 3 we give the explicit form of the enumeration principle for the simplest interesting case, viz. the simple groups of order $\frac{1}{2}p(p^2-1)$ where p is a prime greater than 3. The alternating group on six symbols is included for comparison.

Finally, in § 4, we give a selection of the actual numerical values of the invariants which the formulae allow us to calculate. These could for the most part be obtained directly, the advantage of the inversion formulae being merely that they enable us to handle a large number of special cases uniformly and with economy of thought.

1. The Invariants ϕ_F and d_F

1.1. By an *n-basis* of a group G we mean any ordered set of n elements $X_1, X_2, \ldots, X_n$ of G which generate G:

$$G = \{X_1, X_2, \ldots, X_n\}.$$

* Given also by L. Weisner in the paper referred to at the beginning of § 2.

† J. J. Sylvester, *Comptes rendus*, 96 (1883), 463, for the binomial inversion formula; E. Landau, *Zahlentheorie* (1), 22, Satz 38, for the Möbius formula.

Two n-bases $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are the same, if and only if $X_i = Y_i$ for each $i = 1, 2, \ldots, n$. The members of an n-basis need not be all distinct.

We shall denote by $\phi_n(G)$ the total number of distinct n-bases of G, and call ϕ_n the *nth Eulerian function*. If G cannot be generated by so few as n elements, then $\phi_n(G) = 0$.

If G is cyclic of order m, then $\phi_1(G) = \phi(m)$, where $\phi(m)$ is the Eulerian function of arithmetic.

1.2. Besides the function $\phi_n(G)$, we shall also consider the number of n-bases $X_1, \ldots, X_n$ of G for which the X_i satisfy certain prescribed relations

$$f(X) = g(X) = \ldots = 1, \tag{1.2}$$

where $f(X) \equiv f(X_1, X_2, \ldots, X_n)$, $g(X), \ldots$ are given *words** in the n symbols X_i and their inverses.

If F is the group defined by these relations, we denote the number of n-bases of G which satisfy them by $\phi_F(G)$. Thus $\phi_n(G) = \phi_{F_n}(G)$ where F_n is the free group with n generators. The bases themselves may be spoken of as F-bases of G, in this case.

1.3. Two n-bases $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ of G will be called *equivalent*, if there exists an automorphism θ of G which transforms the one into the other:

$$X_i \theta = Y_i,$$

for each $i = 1, 2, \ldots, n$. Otherwise the two bases will be called inequivalent.

Evidently no automorphism of G other than the identity can transform any basis of G into itself. Thus *the group of automorphisms $A = A(G)$ of G permutes the various bases of G in accordance with a regular permutation group.*

Hence, *if the order of A be denoted by $a = a(G)$, every class of equivalent bases of G consists of exactly a members.*

If one member of a class of equivalent bases is an F-basis, then every member of that class will be an F-basis. Therefore

the invariants $\phi_F(G)$ are all multiples of $a(G)$.

Hence we may write

$$\phi_F(G) = a(G) d_F(G). \tag{1.3}$$

* That is, finite products of the symbols X_i and X_i^{-1}, the order of the factors in a given *word* being (of course) prescribed but arbitrary, while an arbitrary number of occurrences of any symbol is allowed.

1.4. The new invariants $d_F(G)$ thus introduced have an important significance, which we shall now consider.

DEFINITION. A self-conjugate sub-group D of F such that $F/D \cong G$ will be called a *G-defining sub-group* of F.

If $\phi(X)$, $\psi(X)$,... is any set of elements of D which with their transforms under F suffice to generate D, then the equations

$$f(X) = g(X) = ... = 1$$

and

$$\phi(X) = \psi(X) = ... = 1$$

can be taken as a system of defining relations for G. Such a system of defining relations we may call an *F-definition* of G. The choice of the words $\phi(X)$,... from D is to a large extent arbitrary. However, we shall say that the given F-definition of G *belongs to* the defining sub-group D; and also that two F-definitions of G are *equivalent* or not according as they belong to the same defining sub-group or different defining sub-groups.

Thus the two 2-definitions (F being the free group generated by X_1 and X_2)

$$X_1^2 = X_2^3 = (X_1 X_2)^5 = 1,$$

and

$$X_1^3 = X_2^2 = (X_2 X_1)^5 = 1$$

of the icosahedral group are inequivalent, because they belong to different defining sub-groups, in spite of the fact that one is derived from the other merely by interchanging X_1 and X_2. Indeed two n-definitions of G are equivalent, if and only if the system of all relations satisfied by the n generators is precisely the same in the two cases, regard being given to the order of the generators.

To every class of equivalent F-bases $X_1, X_2, ..., X_n$ of G there corresponds a uniquely-determined G-defining sub-group D of F, viz. that formed by all the elements $\phi(X)$ of F for which $\phi(X) = 1$ in G. Conversely, any such D determines a class of equivalent F-definitions: and, if $X_1, ..., X_n$ and $Y_1, ..., Y_n$ are any two n-bases of G which satisfy the defining relations concerned, then clearly $X_1\theta = Y_1, ..., X_n\theta = Y_n$ defines an automorphism θ of G and the two bases are equivalent. Hence we have

THEOREM 1.4. *There is a* (1.1) *correspondence between the classes of equivalent F-bases of G and the G-defining sub-groups of F; $d_F(G)$ is the number of distinct self-conjugate sub-groups D of F which give $F/D \cong G$.*

1.5. Thus it is seen that $\phi_F(G)$ and $d_F(G)$ do not depend on the particular set of relations (1.2) taken to define F but only on the groups G and F themselves. The concepts of F-basis and F-definition, on the other hand, clearly do depend on the particular definition of F chosen, and this must therefore be indicated if these terms are not to be ambiguous. In the most important case we shall consider, that of the free group F_n with n generators, it is clear that we shall mean the definition of F_n with n generators and no relations.

1.6. The meet of all the G-defining sub-groups of F is a characteristic sub-group of F depending only on G: we denote it by $G(F)$. Thus by a study of the F-definitions of G we obtain information about the structure of F.

If G is a simple group of composite order, then the meet $G(F)$ is a direct meet and

$F/G(F)$ is the direct product of exactly $d_F(G)$ groups isomorphic with G.

In this case, then, we have an alternative definition of the number $d_F(G)$: it is the greatest number d for which F is homomorphic to a direct product of d groups G. In particular, if $F = F_n$, we have that *$d_n(G)$ is the greatest number d for which the direct product of d groups isomorphic with G can be generated by n elements.* For example, if G is the icosahedral group, we shall see that $d_2(G) = 19$; this means that the direct product of nineteen icosahedrals can be generated by two elements, but not the direct product of twenty.

We pass next to the problem of *calculating* these invariants ϕ_F and d_F, in any particular case.

2. The Enumeration Principle*

2.1. Let S be any finite system of sub-sets of a given set G, and let G itself belong to S. Let $f(H)$ be any function defined for all

* (Added Jan. 23, 1936.)

My attention has been drawn to the paper, 'Abstract Theory of Inversion of Finite Series', *Trans. American Math. Soc.* 38 (1935), 474–84, by Louis Weisner, with which § 2 of the present paper is closely related. As the reader who consults both papers will easily see, Weisner's function $\mu(x_1/x_2)$, defined in any *hierarchy*, is essentially the same as the Möbius function of (2.1). The differences are purely superficial (and partly notational), the most obvious being that Weisner works with an abstractly defined relation '/' (*divides*), while I use the set-theoretic relation '$\geqslant$' (*contains*). To pass from one theory to the other, one must replace each a of Weisner's hierarchy by the set of all

$H \in S$, and let $g(H)$ be the summatory function derived from f, viz.

$$g(H) = \sum_{K \leqslant H} f(K), \tag{2.11}$$

the sum being taken over all members K of S which are contained in H.

Then we have
$$f(G) = \sum_{H} \mu_S(H) g(H), \tag{2.12}$$

the sum being taken over all members H of S; where the *Möbius function* μ_S is defined by the equations

$$\mu_S(G) = 1, \tag{2.13}$$

and
$$\sum_{K \geqslant H} \mu_S(K) = 0 \tag{2.14}$$

for all $H < G$, the sum being taken over all K of S which contain H.

It is easy to see (e.g. by induction over the relation 'contains') that equations (2.13) and (2.14) are compatible and suffice to determine the function μ_S uniquely. Substituting the value of $g(H)$ into the right-hand side of (2.12), we obtain as coefficient of $f(H)$ precisely

$$\sum_{K \geqslant H} \mu_S(K).$$

By equations (2.13) and (2.14) this vanishes except for $H = G$, when it is 1. Thus the formula is proved.

2.2. For the purpose of calculating $\mu_S(H)$, the equations (2.14) are particularly convenient. However, it is easy to give an explicit expression for this function, viz.

$$\mu_S(H) = \lambda_S(H) - \lambda'_S(H) + \lambda''_S(H) - \ldots, \tag{2.21}$$

where $\lambda_S^{(n)}(H)$ is the number of distinct *chains* of members of S,

$$G = K_0 > K_1 > \ldots > K_n = H, \tag{2.22}$$

of length n, which can be stretched from G to H. $\lambda_S(H)$ is 0 for $H < G$, and 1 for $H = G$.

To show that the function defined by (2.21) satisfies equation (2.14), we need only associate each chain (2.22) from G to H with the chain
$$G = K_0 > K_1 > \ldots > K_{n-1} = K$$

multiples of a, just as in the step from numbers to principal ideals. The version given here seems somewhat the more general, in that my enumeration principle is independent of the existence of *meet* (i.e. l.c.m.) and *join* (i.e. g.c.d.), while these are assumed to exist in Weisner's theory (axioms 4 and 5 for hierarchies). I should like to take this opportunity of acknowledging Weisner's priority.

from G to K, where $K = K_{n-1} > H$. We have thus a (1, 1) correspondence between the chains from G to H on the one hand and the chains from G to K with $K > H$ on the other, such that corresponding chains contribute opposite signs, $(-1)^n$ and $(-1)^{n-1}$, to the sum

$$\sum_{K \geqslant H} \mu_S(K).$$

Thus this sum must vanish.

2.3. In what follows we are mainly interested in the case in which S consists of all the sub-groups of a group G of finite order. In this case we may write μ_G in place of μ_S, and shall call $\mu_G(H)$ the ***Möbius function*** of G. $\mu_G(H)$ is defined for all sub-groups H of G.

In this case the system S is somewhat special and forms a *lattice*, to use the convenient term introduced by Garrett Birkhoff.* Any given set of members of S has a unique *meet* (which coincides with their common part) and a unique *join* (which does not as a rule coincide with their set-theoretic sum). Accordingly, in calculating the Möbius function of a group we shall find the following rule very useful:

THEOREM 2.3. *If meets exist in S, then $\mu_S(H)$ can differ from* 0 *only if $H = G$, or if H is the meet of a certain number of maximal members of S.*

By the meet of any given set T of members of S we mean here a member M of S with the properties:

(i) M is contained in every member of T, and (ii) every member of S with the property (i) is contained in M.

Thus the meet is not necessarily the common part in the set-theoretic sense. By saying that meets exist in S we mean that every set T of members of S has a meet in the sense just defined. (The meet of the null set is G.)

A member K of S is said to be *maximal* if $K < G$ and if there is no member L of S for which $K < L < G$.

Theorem (2.3) may be proved by induction over the relation 'contains'. Suppose that $H < G$, and that H is not the meet of any set of maximal members of S; and suppose the result proved for all members $K > H$ with this property, so that $\mu_S(K) = 0$ for all such K. Let M be the meet of all the maximal members of S which contain H. Then we have $H < M$, and

$$-\mu_S(H) = \sum_{K > H} \mu_S(K);$$

* Garrett Birkhoff, *Proc. Cambridge Phil. Soc.* 29 (1933), 441–64.

and in this sum all the terms vanish for which $K \not\geqslant M$, since such a K cannot be a meet of maximal members of S. Thus

$$-\mu_S(H) = \sum_{K \geqslant M} \mu_S(K),$$

and this vanishes, since $M < G$. Hence $\mu_S(H) = 0$ and the result follows generally by induction.

2.4. *Duality.* If S is any finite system of sets all of which contain a given set E, and if E itself belongs to S, then the duality between the relations '$\geqslant$' and '$\leqslant$' gives rise to another enumeration principle whose enunciation differs from that of section (2.1) only in having G replaced by E and all the inequalities reversed. To avoid confusion we may denote the Möbius function in this case by $\bar{\mu}_S$: it is defined by the equations

$$\bar{\mu}_S(E) = 1, \tag{2.41}$$

and

$$\sum_{K \leqslant H} \bar{\mu}_S(K) = 0 \tag{2.42}$$

for all $H > E$, the sum being taken over all K of S which are contained in H; and we have corresponding to (2.21),

$$\bar{\mu}_S(K) = \bar{\lambda}_S(K) - \bar{\lambda}'_S(K) + \bar{\lambda}''_S(K) - \dots, \tag{2.43}$$

where $\bar{\lambda}_S^{(n)}(K)$ is the number of chains of members of S from K to E which are of length precisely n.

If S consists of all the sub-groups of a group G of finite order, we may take E to be the identity and accordingly in this case we shall write $\bar{\mu}_1$ for $\bar{\mu}_S$; and a comparison of (2.21) with (2.43) gives at once the duality relation

$$\bar{\mu}_1(G) = \mu_G(1), \tag{2.44}$$

which provides a useful check on the accuracy of our calculations.

For our present problem, however, the conjugate Möbius function $\bar{\mu}_1(H)$ is of less importance than $\mu_G(H)$. Here we merely note the dual of Theorem (2.3):

If joins exist in S, then $\bar{\mu}_E(H)$ can differ from 0, *only if $H = E$ or if H is the join of a certain number of minimal members of S.*

Here E is the join of the null set. The terms 'join' and 'minimal' are understood to be the duals of the terms 'meet' and 'maximal' as previously defined.

2.5. Three particular cases of the enumeration principle are known, and probably others also.

(1) S consists of all the 2^n sub-sets of a set of n things. If H is any one of these sub-sets, containing r of the n things, then

$$\mu_S(H) = (-1)^{n-r},$$

as one can see at once from the fact that, with $k = n-r > 0$,

$$1-k+\binom{k}{2}-\binom{k}{3}+\ldots = 0,$$

which is precisely equation (2.14), since there are just $\binom{k}{s}$ sets containing H which have $n-s$ members.

In this case the enumeration principle appears in very many guises in various parts of mathematics, e.g. in the Poincaré formula of the theory of probability, or in the note of Sylvester already referred to.

The lattice S may be called a *Boolean lattice* in this case.

2.6. (2) S consists of the sub-groups other than 1 of an *infinite cyclical group* G. Here S is (strictly speaking) infinite and in applying the principle we must accordingly restrict ourselves to those members of S which contain a fixed sub-group G_n of finite index n. However, it is obvious that μ_S is independent of this restriction; and in fact we have in this case

$$\mu_S(G_n) = \mu(n),$$

where $\mu(n)$ is the Möbius function of the elementary theory of numbers, given by the rule that $\mu(n) = (-1)^r$ if n is the product of r distinct primes ($r = 0, 1, 2,\ldots$), and $\mu(n) = 0$ if n is divisible by the square of a prime.

That this is so is easily seen from (2.3) together with the preceding example. For the maximal members of S are simply the sub-groups G_p where p is a prime divisor of n. Thus G_n is a meet of maximal members of S only if n is square-free; and if this is so, if for instance n is the product of r distinct primes, the lattice S is isomorphic with the Boolean lattice of example (1) consisting of all the sub-sets of r things.

In this case (2.12) is the Möbius inversion formula of the elementary theory of numbers (cf. Landau, loc. cit.).

2.7. (3) S consists of the sub-groups of *a group G of order p^n, where p is a prime.* In this case the enumeration principle was given (in a less general but essentially equivalent form) in the paper already cited in the introduction.

In the case of a prime-power group the maximal sub-groups of G are all self-conjugate, and their meet D is such that G/D is an elementary Abelian group. Thus, by Theorem 2.3, $\mu_G(H) = 0$ except when $H \geqslant D$. If H contains D and is of index p^α in G, then (loc. cit.)

$$\mu_G(H) = (-1)^\alpha p^{\frac{1}{2}\alpha(\alpha-1)}. \tag{2.7}$$

2.8. The following result, which we shall not use, is of interest in that it generalizes a well-known property of $\mu(n)$.

If $G = G_1 \times G_2$, where the orders of G_1 and G_2 are co-prime, then every sub-group H of G is uniquely expressible in the form $H = H_1 \times H_2$ with $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$, and

$$\mu_G(H) = \mu_{G_1}(H_1)\mu_{G_2}(H_2).$$

For the function of H defined by this relation in fact satisfies the equations

$$\mu_G(G) = 1, \qquad \sum_{K \geqslant H} \mu_G(K) = 0$$

for $H < G$.

This, together with (2.7), allows us to write down the Möbius function for any group whose Sylow sub-groups are all self-conjugate: in particular, for any Abelian group.

3. The Explicit Inversion Formulae for some Simple Groups

3.1. Supposing that for a given group G the Möbius function μ_G has been found, we may now consider how to calculate the Eulerian functions $\phi_n(G)$ and $\phi_F(G)$, from which the intrinsically more interesting functions $d_n(G)$ and $d_F(G)$ may be derived at once with the help of (1.3).

To find $\phi_F(G)$ by the enumeration principle we have first to calculate for every sub-group H of G the summatory function

$$\sigma_F(H) = \sum_{K \leqslant H} \phi_F(K). \tag{3.11}$$

Evidently $\sigma_F(H)$ *is the total number of solutions $X_1, \ldots, X_n$ of the defining equations of F for which the X_i all lie in H.*

If $F = F_n$, then the summatory function $\sigma_n(H)$ becomes h^n, where h is the order of H. Thus

$$\phi_n(G) = \sum_{H \leqslant G} \mu_G(H) h^n. \tag{3.12}$$

Similarly we may notice that the number of n-bases of G whose terms are all distinct is given by

$$\sum_{H \leqslant G} \mu_G(H) \binom{h}{n}. \tag{3.13}$$

3.2. After the free groups F_n, the next simplest case is when F is the free product of n cyclical groups of orders $a_1, a_2, \ldots, a_n$ respectively. The relations (1.2) are then taken to be

$$X_1^{a_1} = X_2^{a_2} = \ldots = X_n^{a_n} = 1. \tag{3.2}$$

It is not necessary to suppose that the a_i are all finite; if one of the a_i is infinite, the corresponding relation $X_i^{a_i} = 1$ must be omitted; if they are all infinite, we are back in the preceding case.

This choice of F gives for $\phi_F(G)$ the number of n-bases $X_1, \ldots, X_n$ for which the order of X_i divides a_i; and $\sigma_F(T)$ *becomes the product of the n numbers $s_m(T)$ $(m = a_1, a_2, \ldots, a_n)$, where $s_m(T)$ is the number of elements of T whose orders divide m.*

If in place of $s_m(T)$ we take $\overline{s_m}(T)$, the number of elements of T whose orders are equal to m, we obtain the number of n-bases of G with terms whose orders are exactly $a_1, a_2, \ldots, a_n$ respectively. And so on.

3.3. Thus, in all the cases with which we shall deal the calculation of the summatory function is trivial, and we shall consequently pass on to the expression giving ϕ_F in terms of σ_F, viz.

$$\phi_F(G) = \sum_{H \leqslant G} \mu_G(H)\sigma_F(H). \tag{3.31}$$

It will be noticed that the function $\sigma_F(H)$ depends only on the type of the sub-group H and not at all on the relation of H to G.

In general, if $\phi(H)$ is any function defined for the sub-groups H of G and such that $\phi(H)$ depends only on the type of H, and if

$$\sigma(T) = \sum_{K \leqslant T} \phi(K) \tag{3.32}$$

is the corresponding summatory function, we may write

$$\phi(G) = \sum_{i=1}^{k} \nu_G(T_i)\sigma(T_i), \tag{3.33}$$

where the sum is taken over all the distinct types of group,

$$T_1 = G, \qquad T_2, \ldots, T_k = 1, \tag{3.34}$$

which occur as sub-groups of G; and where

$$\nu_G(T) = \sum_{H \cong T} \mu_G(H), \tag{3.35}$$

this sum being taken over all the sub-groups H of G which are isomorphic with T.

This concise form (3.33) is most convenient for calculation, and we shall refer to it as *the inversion formula for the group G.*

The function ν_G is easily found from the Möbius function; for it

is evident that $\mu_G(H) = \mu_G(H')$ if H and H' are any two conjugate sub-groups of G. Thus, if there are t distinct classes of conjugate sub-groups $C_1,..., C_t$ of G which are isomorphic with T, and if the corresponding values of μ_G are $\mu_1,..., \mu_t$, we have

$$\nu_G(T) = c_1\mu_1+c_2\mu_2+...+c_t\mu_t, \tag{3.36}$$

where c_i is the number of sub-groups which belong to the class C_i. (In most of the cases we need to consider, t will be 1.)

3.4. We shall now give the explicit form of the inversion formula (3.33) in a number of simple cases. The essential step is always the calculation of the Möbius function μ_G: and throughout the following we shall be making continuous use of the formulae (2.14), which give the value of $\mu(H)$ if the values of $\mu(K)$ for all $K > H$ are already known.

The two most obvious and frequent cases are contained in the following two rules:

(3.41) *If H is a maximal sub-group of G, then* $\mu(H) = -1$.

(3.42) *If $H < G$ is not itself a maximal sub-group but is the meet of any two distinct maximal sub-groups which contain it, then* $\mu(H) = m-1$, *where m is the number of distinct maximal sub-groups of G which contain H.*

These two rules, together with Theorem (2.3), yield the major part of the values of μ_G in the cases we shall consider.

3.5. *Notation.* In what follows we denote by E_q an elementary group of order q: thus E_4 is the *Vierergruppe*. And O_8, T_{12}, O_{24}, and I_{60} mean, respectively, the octic group, the tetrahedral group, the octahedral group, and the icosahedral group. Further, C_h and D_{2h} are the cyclic group of order h and the dihedral group of order $2h$; thus $D_8 = O_8$. Finally, the group of order qk, where $q = p^n$ and k divides p^n-1, which is obtained by extending an E_q by a regular automorphism of order k, will be denoted by $M_{q,k}$: its elements may be identified with the transformations

$$x' = \lambda x+\mu,$$

where λ, μ belong to the field $GF(q)$ of Galois imaginaries of order q, and $\lambda^k = 1$; here p is any prime. Thus $M_{3,2} = D_6$, the symmetric group on three symbols; $M_{4,3} = T_{12}$; for $q = p$, $n = 1$, we have the metacyclic group $M_{p,k}$ of order pk.

3.61. The sub-groups of $M_{p,k}$ are: one $M_{p,h}$ and (if $h > 1$) p cyclical groups C_h for each divisor h of k, and also the identity. Hence the Möbius function of $M_{p,k}$ is given by

$$\mu(M_{p,h}) = \mu(h') \quad (hh' = k),$$

$$\mu(C_h) = -\mu(h') \quad (h > 1),$$

and

$$\mu(1) = -p\mu(k).$$

Thus the inversion formula is

$$\phi(M_{p,k}) = \sum_{h|k} \mu(h')[\sigma(M_{p,h}) - p\sigma(C_h)].$$

Here $\mu(n)$ is the ordinary Möbius function.

3.62. The maximal sub-groups of T_{12} are: one E_4 and four C_3. The meet of any two of these is 1. Hence $\mu(1) = 4$ and the inversion formula is:

$$\phi(T_{12}) = \sigma(T_{12}) - \sigma(E_4) - 4\sigma(C_3) + 4\sigma(1).$$

3.63. The maximal sub-groups of O_{24} are: one T_{12}, three O_8 and four D_6. Of the sub-groups E_4, only the self-conjugate one is a meet of maximals, and this lies in the T_{12} and the three O_8: thus $\nu(E_4) = 3$. The three C_4 are not meets of maximals: thus $\nu(C_4) = 0$. The four C_3 each lie in two maximals, and so $\nu(C_3) = 4$. Finally, of the C_2, those in T_{12} are not meets of maximals; while the other six each lie in three maximals viz. one O_8 and two D_6: thus $\nu(C_2) = 12$, whence $\nu(1) = \mu(1)$ is found, from the formula

$$\sum \nu(T_i) = 0,$$

to be -12. The inversion formula is therefore

$$\phi(O_{24}) = \sigma(O_{24}) - \sigma(T_{12}) - 3\sigma(O_8) - 4\sigma(D_6) + \\ + 3\sigma(E_4) + 4\sigma(C_3) + 12\sigma(C_2) - 12\sigma(1).$$

3.64. In an exactly similar way, we find for the icosahedral group:

$$\phi(I_{60}) = \sigma(I_{60}) - 5\sigma(T_{12}) - 6\sigma(D_{10}) - 10\sigma(D_6) + \\ + 20\sigma(C_3) + 60\sigma(C_2) - 60\sigma(1).$$

For here, the maximal sub-groups are: five T_{12}, six D_{10} and ten D_6; and, apart from the identity, their only meets are the ten C_3 (each of which lies in one D_6 and two T_{12}) and the fifteen C_2 (each of which lies in one T_{12}, two D_{10}, and two D_6). Thus $\mu(C_3) = 2$ and $\mu(C_2) = 4$; whence we deduce $\mu(1) = -60$, and the inversion formula follows.

3.7. We denote by B_{24}, B_{48}, and B_{120} the so-called binary tetrahedral, octahedral, and icosahedral groups. These have a central of

order 2 whose factor group is T_{12}, O_{24}, and I_{60}, respectively. Since every maximal sub-group contains the central, the inversion formulae for these three groups may be written down at once from that of the corresponding factor group. They are

$$\phi(B_{24}) = \sigma(B_{24}) - \sigma(Q_8) - 4\sigma(C_6) + 4\sigma(C_2);$$

$$\phi(B_{48}) = \sigma(B_{48}) - \sigma(B_{24}) - 3\sigma(Q_{16}) - 4\sigma(D_{12}) + 3\sigma(Q_8) + \\ + 4\sigma(C_6) + 12\sigma(E_4) - 12\sigma(C_2);$$

$$\phi(B_{120}) = \sigma(B_{120}) - 5\sigma(B_{24}) - 6\sigma(D'_{20}) - 10\sigma(D'_{12}) + 20\sigma(C_6) + \\ + 60\sigma(C_4) - 60\sigma(C_2).$$

Here Q_8 is the quaternion group, Q_{16} the 2-Sylow sub-group of B_{48}, D'_{12} and D'_{20} groups obtained by extending a C_6 and C_{10} (respectively) by an automorphism of order 2, differing from D_{12} and D_{20} in having their 2-Sylow sub-groups cyclical and not elementary.

3.8. We consider next the simple group of order 168. The maximal sub-groups are: fourteen O_{24} and eight $M_{7,3}$. Each of the twenty-one O_8 and the twenty-eight D_6 lies in two of the O_{24}: thus

$$\mu(O_8) = \mu(D_6) = 1.$$

Also $\mu(T_{12}) = \mu(C_7) = 0$, since the T_{12} and C_7 are not meets of maximals. Of the fourteen E_4, each lies in four O_{24} and three O_8, so that $\mu(E_4) = 0$. Similarly $\mu(C_4) = 0$. The twenty-eight C_3 each lie in two $M_{7,3}$, and all the other sub-groups which contain any one of them also contain its normalizing D_6: thus $\mu(C_3) = 2$. Of the twenty-one C_2, each lies in six O_{24}, five O_8, four D_6: thus $\mu(C_2) = -4$. Finally, we have

$$\mu(1) = -1+14+8-21-28-56+84 = 0.$$

Hence we have the inversion formula

$$\phi(G_{168}) = \sigma(G_{168}) - 14\sigma(O_{24}) - 8\sigma(M_{7,3}) + 21\sigma(O_8) + \\ + 28\sigma(D_6) + 56\sigma(C_3) - 84\sigma(C_2).$$

For the simple group of order 360, i.e. the alternating group on six symbols, the calculations are slightly more complicated, and we may quote the result:

$$\phi(G_{360}) = \sigma(G_{360}) - 12\sigma(I_{60}) - 10\sigma(M_{9,4}) - 30\sigma(O_{24}) + \\ + 60\sigma(T_{12}) + 36\sigma(D_{10}) + 45\sigma(O_8) + 240\sigma(D_6) + \\ + 90\sigma(C_4) - 240\sigma(C_3) - 900\sigma(C_2) + 720\sigma(1);$$

while, for the simple group of order 660, we get

$$\phi(G_{660}) = \sigma(G_{660})-22\sigma(I_{60})-12\sigma(M_{11,5})-55\sigma(D_{12})+ \\ +55\sigma(T_{12})+66\sigma(D_{10})+220\sigma(D_6)+132\sigma(C_5)+ \\ +165\sigma(E_4)-220\sigma(C_3)-990\sigma(C_2)+660\sigma(1).$$

3.9. We may now proceed to consider in detail *the simple groups of order* $\frac{1}{2}p(p^2-1)$, *where* p *is a prime exceeding* 3. *This we denote by* M^p. Thus M^5, M^7, and M^{11} are the icosahedral group and the groups of order 168 and 660, respectively. These are the three exceptional groups of Galois, and for that reason we have had to treat them separately. If $p > 11$, which we now assume, *four cases must be distinguished*:*

(i) $p \equiv \pm 1 \pmod 5$ *and* $\pm 1 \pmod 8$,

(ii) $p \equiv \pm 1 \pmod 5$ *and* $\pm 3 \pmod 8$,

(iii) $p \equiv \pm 2 \pmod 5$ *and* $\pm 1 \pmod 8$,

(iv) $p \equiv \pm 2 \pmod 5$ *and* $\pm 3 \pmod 8$.

For convenience we write $\frac{1}{2}(p-1) = q$ *and* $\frac{1}{2}(p+1) = r$; *also* $2pqr = g$, *the order of* M^p.

Then it is well known† that M^p has the following sub-groups: $2r$ of the $M_{p,q}$, pq of the D_{2r} and pr of the D_{2q}, these being all maximal; there are also pq cyclical $C_{r'}$ for each divisor r', exceeding unity, of r; and pr cyclical $C_{q'}$ for each divisor q', exceeding unity, of q; further, there are $\frac{1}{2}g/r'$ dihedral $D_{2r'}$ and $\frac{1}{2}g/q'$ dihedral $D_{2q'}$, provided r' and q' are not equal to 2. The number of $D_4 = E_4$ is, however, $\frac{1}{12}g$; and there are also $\frac{1}{12}g$ tetrahedral sub-groups. Finally, if $p \equiv \pm 1 \pmod 5$, i.e. in cases (i) and (ii), there are $\frac{1}{30}g$ icosahedral sub-groups; while if $p \equiv \pm 1 \pmod 8$, i.e. in cases (i) and (iii), there are $\frac{1}{12}g$ octahedral sub-groups.

This list of sub-groups is exhaustive: and since any two sub-groups of the same type, if not conjugate in M^p, are in any case conjugate under its group of automorphisms, there is no difficulty in calculating the values of the Möbius function.

We find $\qquad \mu(M_{p,q}) = \mu(D_{2q}) = \mu(D_{2r}) = -1$

and $\mu(C_q) = 2$. For the other values, we must distinguish between the four cases (i)–(iv)

* The present conventions and notations are preserved to the end of the paper. † Burnside, *Theory of Groups* (2nd ed.), chapter XX.

Case / H	(i)	(ii)	(iii)	(iv)
I_{60}	-1	-1	0	0
O_{24}	-1	0	-1	0
T_{12}	2	1	0	-1
D_{10}	2	2	0	0
O_8	2	0	2	0
D_6	4	2	2	0
E_4	0	3	0	3

Values of $\mu(H)$

Here we have written 0 in those cases in which no sub-group of the kind indicated occurs. There remain only

$$\mu(C_3) = (-2,-1,0,1)\tfrac{2}{3}s,$$

$$\mu(C_2) = (-5,-3,-1,1)t,$$

$$\mu(1) = (2,1,0,-1)g;$$

where s is either q or r according as $p \equiv \pm 1 \pmod 3$, and t is either q or r according as $p \equiv \pm 1 \pmod 4$. All other sub-groups not yet mentioned give $\mu = 0$ in every case.

Hence we have the following inversion formulae:

THEOREM 3.9. *For all p exceeding* 3,

$$\phi(M^p) = \sigma(M^p) - 2r\sigma(M_{p,q}) - pq\sigma(D_{2r}) + pr[2\sigma(C_q) - \sigma(D_{2q})] + gS,$$

where S depends on the particular case to which M^p belongs and is given by the expressions

(i) $-\tfrac{1}{30}\sigma(I_{60}) - \tfrac{1}{12}\sigma(O_{24}) + \tfrac{1}{6}\sigma(T_{12}) + \tfrac{1}{5}\sigma(D_{10}) + \tfrac{1}{4}\sigma(O_8) +$
$+\tfrac{2}{3}\sigma(D_6) - \tfrac{2}{3}\sigma(C_3) - \tfrac{5}{2}\sigma(C_2) + 2\sigma(1),$

(ii) $-\tfrac{1}{30}\sigma(I_{60}) + \tfrac{1}{12}\sigma(T_{12}) + \tfrac{1}{5}\sigma(D_{10}) + \tfrac{1}{3}\sigma(D_6) +$
$+\tfrac{1}{4}\sigma(E_4) - \tfrac{1}{3}\sigma(C_3) - \tfrac{3}{2}\sigma(C_2) + \sigma(1),$

(iii) $-\tfrac{1}{12}\sigma(O_{24}) + \tfrac{1}{4}\sigma(O_8) + \tfrac{1}{3}\sigma(D_6) - \tfrac{1}{2}\sigma(C_2),$

(iv) $-\tfrac{1}{12}\sigma(T_{12}) + \tfrac{1}{4}\sigma(E_4) + \tfrac{1}{3}\sigma(C_3) + \tfrac{1}{2}\sigma(C_2) - \sigma(1).$

It is easily verified that, for $p = 5, 7, 11$, these formulae reduce to the simpler forms noted above. ($I_{60} = M^5$ is considered to belong to case (iv).)

4. Numerical results

4.1. From the inversion formulae the Eulerian functions may be written down as described in (3.3). From ϕ_n or ϕ_F we derive the invariants d_n or d_F by dividing by the order of the group of automorphisms; for the M^p this is $2g$, and for the alternating group on six symbols it is 1440.

We give below a selection from the more interesting numerical results which may be obtained in this way. The number of bases $X_1,\ldots, X_n$ of G for which the order of X_i is exactly m_i may be written

$$a(G)c_{m_1, m_2,\ldots, m_n}(G).$$

We shall consider particularly the invariants $c_{m,n}$ related to the 2-bases (their sum is clearly d_2); and also $c_{2,2,2}$, which is of interest owing to the fact that none of our simple groups can be generated by two elements of order 2, so that $c_{2,2} = 0$. The $c_{m,n}$ are related in an obvious way to the invariants d_F where F is the free product of a cyclical group of order m and one of order n; in fact, if m and n are primes, we have $c_{m,n} = d_F$, at least when G is not cyclical.

4.2. *For $d_2(M^p)$, we have the values*

$$\tfrac{1}{4}(p+1)(p^2-2p-1)-\epsilon,$$

where $\epsilon = 49, 40, 11, 2$, *in the four cases* (i)–(iv).

Thus for $p = 5, 7, 11, 13$, we have for d_2 the values 19, 57, 254, 495, respectively.

Also $$d_2(G_{360}) = 53.$$

This is smaller than $d_2(M^7)$, although the order of G_{360} is more than twice that of M^7: this anomaly is due to the fact that the alternating group on six symbols has three classes of outer automorphisms, while M^7 has only one.

4.3. *For $c_{2,2,2}(M^p)$ we have the values*

$$\tfrac{1}{8}(p+1)(p^2+p-8)-\tfrac{1}{2}\epsilon,$$

or $$\tfrac{1}{8}(p^3-4p^2+p-14)-\tfrac{1}{2}\epsilon,$$

according as $p \equiv 1$ *or* 3 (mod 4); *where* $\epsilon = 97, 71, 21, -5$ *in the four cases* (i)–(iv).

Thus for $p = 5, 7, 11, 13$, the values of $c_{2,2,2}$ are 19, 7, 70, 307, respectively.

Also $$c_{2,2,2}(G_{360}) = 27.$$

4.4. The invariants $c_{m,n}(G)$ are most conveniently given in the form of a table of double entry. Thus for $G = M^5 = I_{60}$, we have the table

$m \backslash n$	2	3	5
2	0	1	2
3	1	1	4
5	2	4	4

The sum of the nine entries is, rightly, 19, the value of d_2.

Similarly, for the alternating group of order 360 we have $d_2 = 53$ and the table of the $c_{m,n}$ is

$m \backslash n$	2	3	4	5
2	0	0	1	2
3	0	2	3	4
4	1	3	3	9
5	2	4	9	10

For the groups M^p the general results given below show clearly the 'Eulerian' character of the $c_{m,n}$; we use $\phi(m)$ for the ordinary Eulerian function.

For primes $p = 5, 7, 11, 13,\ldots$, the invariants $c_{m,n}$ of the groups M^p are given by the following rules:

$$c_{p,p} = \phi(p) = p-1,$$

$$c_{p,m} = c_{m,p} = \tfrac{1}{2}\phi(m)\phi(p), \text{ if } m \text{ divides } q \text{ or } r.$$

We say that two divisors m and n of q or r are *similar* or *dissimilar* according as they both divide the same one of the two numbers q and r or not. Then

$$c_{m,n} = \tfrac{1}{4}(p-2)\phi(m)\phi(n) - \epsilon_{m,n},$$

if m and n are similar (in particular, if they are not co-prime); while, if m and n are dissimilar,

$$c_{m,n} = \tfrac{1}{4}p\phi(m)\phi(n) - \epsilon_{m,n};$$

where $\epsilon_{m,n} = 0$ with the following exceptions:

$$\epsilon_{2,2} = \tfrac{1}{4}(p-2), \text{ so that } c_{2,2} = 0;$$

$$\epsilon_{2,3} = \epsilon_{3,2} = \tfrac{9}{2}, \tfrac{7}{2}, \tfrac{5}{2}, \tfrac{3}{2} \text{ in the four cases (i)–(iv);}$$

$$\epsilon_{3,3} = 4, 4, 2, 2 \text{ in the four cases;}$$

$$\epsilon_{2,m} = \epsilon_{m,2} = \tfrac{1}{4}\phi(m), \text{ if } m > 5;$$

if M^p has elements of order 4, then

$$\epsilon_{4,4} = 1; \quad \epsilon_{2,4} = \epsilon_{4,2} = \tfrac{3}{2} \quad \text{and} \quad \epsilon_{3,4} = \epsilon_{4,3} = 2;$$

if M^p has elements of order 5 and $p > 5$, then

$$\epsilon_{5,5} = 8; \quad \epsilon_{2,5} = \epsilon_{5,2} = 5 \quad \text{and} \quad \epsilon_{3,5} = \epsilon_{5,3} = 8.$$

We suppose throughout that $m, n > 1$. Naturally $c_{1,n} = c_{n,1} = 0$.

The reader will easily verify that the sum of all the relevant values of the $c_{m,n}$ is in every case equal to d_2.

ON AN ASYMPTOTIC EXPRESSION FOR A CERTAIN INTEGRAL

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IN an investigation of certain real power-series, Le Roy* has obtained asymptotic expressions for the integral

$$J = \int\limits_0^{+\infty} \exp[\alpha x - \phi(x)]\, dx$$

as $\alpha \to +\infty$, where $\phi(x)$ and its first and second derivatives are monotone and continuous for positive values of x and satisfy various further conditions. If, in particular, ϕ, ϕ', ϕ'' become infinite with x, it is stated† that

$$J \sim \exp[\xi\phi'(\xi) - \phi(\xi)]\sqrt{\{2\pi/\phi''(\xi)\}} \qquad (\alpha \to +\infty),$$

where $\phi'(\xi) = \alpha$, the proof of this expression being referred to that given for the case in which $\phi' \to +0$ and $-\phi'' \to +0$, as $x \to +\infty$. It turns out, however, that the fact that $|\phi''|$ is increasing instead of decreasing introduces further difficulties so serious as to render the previous method of proof unsatisfactory. In fact, an example constructed at the end of the present paper shows that the formula itself need not be valid unless further conditions are imposed on ϕ. It is the purpose of the present note to give an exact proof of the foregoing formula under assumptions which imply its validity. The additional assumptions concern the existence and behaviour of the third derivative of ϕ. The necessity for these further assumptions‡ is natural in view of the fact that what is missing in Le Roy's proof is a theorem, Tauberian in nature, concerning the differentiation of asymptotic inequalities.§

* É. Le Roy, *Bull. des Sciences Math.* (2) 24 (1900), 245–68. For further references and applications, cf. G. H. Hardy, *Orders of Infinity* (Cambridge Mathematical Tract, No. 12, 2nd ed.), p. 57.

† É. Le Roy, loc. cit. 263–4, the proof being referred to that of § 4.

‡ Actually, one might dispense with the existence of the third derivative by imposing corresponding restrictions on the difference-quotient of the second derivative.

§ Cf. G. H. Hardy, op. cit., Chap. 5.

We shall, accordingly, obtain an asymptotic expression for integrals of the form

$$J = \int_0^{+\infty} \exp[\alpha x - \phi(x)]\, dx, \tag{1}$$

as $\alpha \to +\infty$, under the assumptions

(i) ϕ, ϕ', ϕ'' are, for $x \geqslant 0$, non-negative, continuous, and monotone increasing, and all three tend to infinity with x;

(ii) ϕ''' exists and is positive if x is positive;

(iii) ϕ'' and ϕ''' are such that ϕ'''/ϕ'' and ψ/ϕ'' tend to definite limits (including $\pm\infty$) as x becomes infinite,* where $\psi = \psi(x)$ is an L-function.†

For a fixed α, the integrand has a maximum at $x = \xi$, where ξ is determined by $\alpha = \phi'(\xi)$, so that ξ is unique by (i). It is clear from (i) that ξ becomes positively infinite with α, and conversely. We then put

$$J = J_1 + J_2 + J_3,$$

where $\quad J_1 = \int_0^{\xi-\epsilon}, \quad J_2 = \int_{\xi-\epsilon}^{\xi+\epsilon}, \quad J_3 = \int_{\xi+\epsilon}^{+\infty}, \quad$ and $\quad \epsilon = \epsilon(\xi)$.

It will appear that‡ $\epsilon(\xi) \prec \xi$. If in J_2 we set $\alpha = \phi'(\xi)$ and make use of the relation

$$\phi(x) = \phi(\xi) + (x-\xi)\phi'(\xi) + \tfrac{1}{2}(x-\xi)^2\phi''(X), \tag{2}$$

where $\xi - \epsilon < X < \xi + \epsilon$, we obtain

$$J_2 \exp[\phi(\xi) - \xi\phi'(\xi)] = \int_{\xi-\epsilon}^{\xi+\epsilon} \exp[-\tfrac{1}{2}(x-\xi)^2\phi''(X)]\, dx.$$

Hence, in view of (i),

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \exp[-\tfrac{1}{2}(x-\xi)^2\phi''(\xi+\epsilon)]\, dx \leqslant J_2 \exp[\phi(\xi) - \xi\phi'(\xi)]$$

$$\leqslant \int_{\xi-\epsilon}^{\xi+\epsilon} \exp[-\tfrac{1}{2}(x-\xi)^2\phi''(\xi-\epsilon)]\, dx.$$

Placing $\quad y = (x-\xi)\sqrt{\{\tfrac{1}{2}\phi''(\xi+\epsilon)\}}\quad$ in the former integral

and $\quad y = (x-\xi)\sqrt{\{\tfrac{1}{2}\phi''(\xi-\epsilon)\}}\quad$ in the latter,

* (iii) implies that the ratios ϕ''/ϕ' and ϕ'/ϕ likewise tend to definite limits. Cf. G. H. Hardy, op. cit. 33–4.

† Actually, the only L-function of which we make use is $\log x$, but this implies the truth of the statement for any L-function. Cf. G. H. Hardy, op. cit. 17, Theorem 13.

‡ For the notation, cf. G. H. Hardy, op. cit., Introduction.

we have

$$(1-\eta)\sqrt{\{2\pi/\phi''(\xi+\epsilon)\}} \leqslant J_2\exp[\phi(\xi)-\xi\phi'(\xi)] \leqslant \sqrt{\{2\pi/\phi''(\xi-\epsilon)\}}, \quad (3)$$

where the positive number η is arbitrarily small for sufficiently large ξ, provided that $\epsilon(\xi)\sqrt{\{\phi''[\xi-\epsilon(\xi)]\}}$ tends with ξ to $+\infty$, in which case the same is necessarily true of $\epsilon(\xi)\sqrt{\{\phi''[\xi+\epsilon(\xi)]\}}$.

Accordingly, if $\epsilon = \epsilon(\xi)$ is such that

$$\epsilon(\xi)\sqrt{\{\phi''(\xi)\}} \succ 1 \quad (4)$$

and

$$\phi''(\xi\pm\epsilon) \sim \phi''(\xi), \quad (5)$$

then (3) implies $J_2 \sim \exp[\xi\phi'(\xi)-\phi(\xi)]\sqrt{\{2\pi/\phi''(\xi)\}}$. (6)

In view of the assumptions (ii) and (iii), a sufficient condition to ensure (5) is*

$$\epsilon(\xi) \prec \frac{\phi''(\xi)}{\phi'''(\xi)}. \quad (7)$$

It will be shown later that $\epsilon(\xi)$ may always be so assigned that both (4) and (7) are simultaneously fulfilled.

Next consider J_1/J_2. From (i) and the definition of J_1, it is seen that

$$J_1 < (\xi-\epsilon)\exp[(\xi-\epsilon)\phi'(\xi)-\phi(\xi-\epsilon)];$$

hence, using (2) for $x = \xi-\epsilon$,

$$J_1 < (\xi-\epsilon)\exp[\xi\phi'(\xi)-\phi(\xi)-\tfrac{1}{2}\epsilon^2\phi''(\xi-\epsilon)].$$

If we combine this with the first of the inequalities (3), we see from (5) that, for sufficiently large ξ,

$$\begin{aligned} J_1/J_2 &< \frac{(\xi-\epsilon)}{\sqrt{(2\pi)(1-\eta)}}\sqrt{\{\phi''(\xi+\epsilon)\}}\exp[-\tfrac{1}{2}\epsilon^2\phi''(\xi-\epsilon)] \\ &< \xi\sqrt{\{\phi''(\xi)\}}\exp[-\tfrac{1}{4}\epsilon^2\phi''(\xi)], \end{aligned} \quad (8)$$

so $J_1 \prec J_2$, provided $\exp[\log(\xi\sqrt{\{\phi''(\xi)\}})] \prec \exp[\tfrac{1}{4}\epsilon^2\phi''(\xi)]$. For the latter it is sufficient that

$$\sqrt{\frac{\log\xi}{\phi''(\xi)}} \prec \epsilon(\xi) \quad (9)$$

and

$$\sqrt{\frac{\log\phi''(\xi)}{\phi''(\xi)}} \prec \epsilon(\xi). \quad (10)$$

Now (4) is certainly satisfied if (9) is, so we have to examine only the compatibility of (7), (9), and (10).

Comparing (7) and (9), we obtain

$$\frac{\phi'''(\xi)}{[\phi''(\xi)]^{\frac{3}{2}}} \prec \frac{1}{\sqrt{(\log\xi)}}. \quad (11)$$

* Cf. G. H. Hardy, op. cit. 41, Theorem 31.

Similarly, from (7) and (10),

$$\frac{\phi'''(\xi)}{[\log\phi''(\xi)]^{-\frac{1}{2}}[\phi''(\xi)]^{\frac{3}{2}}} \prec 1,$$

which is certainly true if

$$\frac{\phi'''(\xi)}{[\phi''(\xi)]^{\frac{3}{2}}} \prec 1. \tag{12}$$

Integration of (11) and (12), together with (i) and the fact that the inequalities obtained by integration can be differentiated in turn by virtue of (iii), shows that (11) and (12) are true. It follows that an $\epsilon = \epsilon(\xi)$ satisfying (4), (7), (9), (10) can always be found under the assumptions (i), (ii), (iii).

Next consider J_3. Setting $x\phi'(\xi)-\phi(x) = -y$, we obtain

$$\int\limits_{-X}^{+\infty} \frac{e^{-y}\,dy}{\phi'(x)-\phi'(\xi)},$$

where $X \equiv [(\xi+\epsilon)\phi'(\xi)-\phi(\xi+\epsilon)]$.

In view of (i),

$$0 < [\phi'(x)-\phi'(\xi)]^{-1} \leqslant [\phi'(\xi+\epsilon)-\phi'(\xi)]^{-1} = [\epsilon(\xi)\phi''\{\xi+\theta\epsilon(\xi)\}]^{-1}$$
$$< [\epsilon(\xi)\phi''(\xi)]^{-1}$$

for all x not less than $\xi+\epsilon$, where $0 < \theta < 1$. Thus

$$0 < \frac{J_3}{J_2} < \frac{1}{\sqrt{(2\pi)(1-\eta)}}\,\frac{\sqrt{\{\phi''(\xi+\epsilon)\}}\exp[-\phi(\xi+\epsilon)+\phi(\xi)+\epsilon(\xi)\phi'(\xi)]}{\epsilon(\xi)\phi''(\xi)};$$

hence, using (2) and (5), if ξ is sufficiently large,

$$0 < \frac{J_3}{J_2} < \frac{1}{\epsilon(\xi)\sqrt{\{\phi''(\xi)\}}}\exp[-\tfrac{1}{2}\{\epsilon(\xi)\}^2\phi''(\xi)], \tag{13}$$

from which it is seen that $J_3 \prec J_2$ in virtue of (4).

It follows that, under the hypotheses (i), (ii), (iii),

$$J \sim \exp[\xi\phi'(\xi)-\phi(\xi)]\sqrt{\{2\pi/\phi''(\xi)\}}. \tag{14}$$

We shall now construct an example of a function $\phi(x)$ such that

(I) condition (i) is satisfied (and condition (ii) could be satisfied also by the introduction of suitable easement curves);

(II) an asymptotic expression for J does not exist.

Let $\phi_1(x)$ be a function satisfying conditions (i), (ii), (iii). In addition, let

$$x < \phi_1''(x) \quad (0 \leqslant x), \tag{15}$$

so that, if $\epsilon(\xi)$ satisfies (10), it satisfies (4) and (9) also. For this ϕ_1, it follows that $J_1 \prec J_2$ and $J_3 \prec J_2$. At the same time, let $\epsilon(\xi)$ be taken

small enough to satisfy (7). It has previously been shown that this is always possible. Moreover, it follows from (10) that we may take $0 < \epsilon(\xi) < 1$, provided ξ is sufficiently large. The corresponding J has then an asymptotic value given by (14).

Next, let $\phi_2''(x) = 2\phi_1''(x)$, so that $\phi_2''(x) > \phi_1''(x)$ if x is positive. Then as $(\log y)/y$ is a monotone decreasing function of y for y exceeding e, we see that an $\epsilon(\xi)$ satisfying (10), hence also (4) and (9), in the case of ϕ_1 will satisfy these conditions in the case of ϕ_2 also. Furthermore, an $\epsilon(\xi)$ satisfying (7) in the case of ϕ_1 will do so likewise in the case of ϕ_2, since

$$\frac{\phi_2''(x)}{\phi_2'''(x)} = \frac{\phi_1''(x)}{\phi_1'''(x)}.$$

Consequently, J formed for ϕ_2 has also the asymptotic value given by (14).

Finally, we form a function ϕ as follows. Let $\phi''(x)$ coincide with $\phi_1''(x)$ up to and including $x = \xi_1+1$, where $\xi_1 > \eta_1+1$, η_1 being sufficiently large for all the foregoing considerations to apply when $x \geqslant \eta_1$. Thence $\phi''(x)$ is represented by a line through $\{\xi_1+1, \phi_1''(\xi_1+1)\}$ and with the slope k_1, where $0 < k_1 < +\infty$ and k_1 is taken so large that this line meets $\phi_2''(x)$ at a point $\{\eta_2, \phi_2''(\eta_2)\}$, say. Let ξ_2 be greater than η_2+1, and let $\phi''(x) = \phi_2''(x)$ between $x = \eta_2$ and $x = \xi_2+1$. Thence let $\phi''(x)$ be represented by a line through $\{\xi_2+1, \phi_2''(\xi_2+1)\}$ with slope k_2, where k_2 is positive but so small that this line meets $\phi_1''(x)$ at a point $\{\eta_3, \phi_1''(\eta_3)\}$, say. Let ξ_3 be greater than η_3+1 and let $\phi''(x) = \phi_1''(x)$ between $x = \eta_3$ and $x = \xi_3+1$. Thence $\phi''(x)$ is represented by a line through

$$\{\xi_3+1, \phi_1''(\xi_3+1)\}$$

having the slope k_3, where $0 < k_3 < +\infty$ and k_3 is so large that the line meets $\phi_2''(x)$ at a point $\{\eta_4, \phi_2''(\eta_4)\}$, say. By continuing this construction indefinitely, one obtains a function $\phi''(x)$, defined for all non-negative x, to which there may be made to correspond a ϕ' and a ϕ. Since $\phi_1'' \leqslant \phi'' \leqslant \phi_2'' = 2\phi_1''$, it follows that

$$\phi_1' \leqslant \phi' \leqslant \phi_2' = 2\phi_1' \quad \text{and} \quad \phi_1 \leqslant \phi \leqslant \phi_2 = 2\phi_1, \tag{16}$$

by a suitable choice of integration constants. In particular, $\phi' = \phi_i'$ and $\phi = \phi_i$ for those values of x for which $\phi'' = \phi_i''$ $(i = 1, 2)$.

Of course, the first two equality signs in each of the foregoing inequalities do not hold simultaneously. Moreover, ϕ satisfies condition (i) for all positive x; and in (ξ_i-1, ξ_i+1) $(i = 1, 2, 3, \ldots)$, it

satisfies conditions (ii) and (iii) also. For any ξ_i, $\exp[x\phi'(\xi_i)-\phi(x)]$ has a maximum at $x = \xi_i$. Hence the computation of J_2 at any ξ_i proceeds precisely as in the case of ϕ_1 or ϕ_2, since we have seen that the positive function $\epsilon(\xi)$ can be taken to be less than 1, if $\xi > \eta_1$. Moreover, as may be seen from the earlier part of the paper, the appraisal of J_1 for a given ξ_i depends only on the value of ϕ in (ξ_i-1, ξ_i+1), and the same is true of J_3. Hence,

$$J_1 \prec J_2 \quad \text{and} \quad J_3 \prec J_2 \quad \text{as} \quad \xi_i \to +\infty,$$

so that $\quad J \sim \exp[\xi_i\,\phi_1'(\xi_i)-\phi_1(\xi_i)]\sqrt{\{2\pi/\phi_1''(\xi_i)\}}\quad (i \text{ odd});$

and $\quad J \sim \exp[\xi_i\,\phi_2'(\xi_i)-\phi_2(\xi_i)]\sqrt{\{2\pi/\phi_2''(\xi_i)\}}\quad (i \text{ even}).$

But

$$\frac{\sqrt{(2\pi/\phi_1'')}\exp[\xi\phi_1'(\xi)-\phi_1(\xi)]}{\sqrt{(2\pi/\phi_2'')}\exp[\xi\phi_2'(\xi)-\phi_2(\xi)]} = \sqrt{\frac{\phi_2''(\xi)}{\phi_1''(\xi)}}\exp[\xi(\phi_1'(\xi)-\phi_2'(\xi))+\phi_2(\xi)-\phi_1(\xi)].$$

In view of (16) this may be written as

$$J_{\phi_1}/J_{\phi_2} \sim \sqrt{2}\exp[-\xi\phi_1'(\xi)+\phi_1(\xi)]. \tag{17}$$

If, therefore, ϕ_1, which has so far been subjected only to conditions (i), (ii), (iii), (15), (16), is such that

$$\phi_1(x) \prec x\phi_1'(x), \quad \text{i.e.} \quad 1/x \prec \phi_1'(x)/\phi_1(x),$$

so that $\log x \prec \log\phi_1(x)$, the ratio (17) approaches zero as $\xi \to +\infty$. From this it is seen that J, formed for ϕ, has no asymptotic formula as $\xi \to +\infty$. This completes the proof.

By the introduction of suitable easement curves in the construction of ϕ'', the function ϕ of the preceding example could be made to have derivatives of arbitrarily high order for all non-negative values of x. From this fact it is clear that the failure of the asymptotic formula under Leroy's conditions is due not to the non-existence of ϕ''' at some points but to the strong oscillation in the rate of increase of ϕ''.

THE THEORY OF ROWLAND'S CONCAVE GRATING

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1. The theory of concave gratings as sometimes presented is apt to convey a confused impression of an instrument acting at once like a mirror (in geometrical optics) and a diffractor (in physical optics). The best presentations avoid this impression, but are none the less fragmentary. Thus nothing is said of what the grating does at points off a certain 'focal curve', and on that curve attention is confined to points of principal maximum intensity. These difficulties are overcome by applying the theory of Kirchhoff's integral to the concave grating, which does not appear to have been done. It is found that the integrals to be evaluated are of the form

$$\int (\cos, \sin)(Lx+Mx^2)\,dx$$

taken over the reflecting portions of the grating. These are identical with those met with in the theory of the plane grating, so that there is a formal correspondence between the two types of grating. $M = 0$ is the equation of the 'focal curve' described by the point of observation, and the intensity at any point on this curve is given by a Fraunhofer formula. The intensity at points off the focal curve is of a lower order of magnitude. The difficulties of its calculation are great, being precisely those which attend the discussion of Fresnel's diffraction phenomena for a plane grating. It appears, therefore, that the manageable problems associated with the concave grating are simple, and easily demarcated.

2. We shall take the grating to be a small part of a circular cylinder. The figure shows the central section which is perpendicular to the generators. R is the radius of the grating and L its end-point. Let x denote distances along the arc LH, and z distances perpendicular to the plane of the paper. Let $Q(\rho_1, i)$ as shown be the source, and $P(\rho, u)$ the point of observation, and let r and r_1 be the distances of P and Q from a point H of the grating.

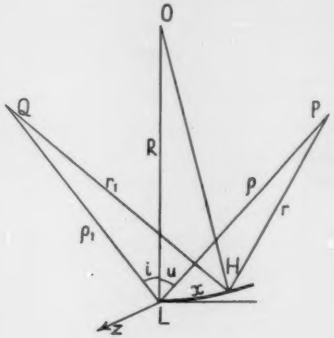
In terms of Kirchhoff's integral the disturbance at P is, to a sufficient degree of accuracy,

$$s_0 = \frac{A(\cos i+\cos u)}{2\lambda\rho\rho_1}\int \sin 2\pi\left(\frac{t}{T}-\frac{r+r_1}{\lambda}\right) dS,$$

where A is a constant, λ, t, T have the usual meanings, and the integral is to be taken over the reflecting parts of the grating.

Since we are not concerned with the aberrations, we need not consider terms of higher order than the second in x and z, and it is easily seen that to this order

$$r+r_1 = \rho+\rho_1+x(\sin i-\sin u)+$$
$$+\tfrac{1}{2}x^2\left[\cos u\left(\frac{\cos u}{\rho}-\frac{1}{R}\right)+\cos i\left(\frac{\cos i}{\rho_1}-\frac{1}{R}\right)\right]+\tfrac{1}{2}z^2\left(\frac{1}{\rho}+\frac{1}{\rho_1}\right).$$



Putting $\quad \dfrac{t}{T} = \dfrac{\rho+\rho_1}{\lambda}+\dfrac{t'}{T}, \qquad A' = \dfrac{A(\cos i+\cos u)}{2\lambda\rho\rho_1},$

we get $\quad s_0 = A'\displaystyle\int \sin\left\{\frac{2\pi t'}{T}-(Lx+Mx^2+Nz^2)\right\}dS,$

where $\quad L = \dfrac{2\pi}{\lambda}(\sin i-\sin u), \qquad (1)$

$$M = \frac{\pi}{\lambda}\left[\cos u\left(\frac{\cos u}{\rho}-\frac{1}{R}\right)+\cos i\left(\frac{\cos i}{\rho_1}-\frac{1}{R}\right)\right], \qquad (2)$$

and $\quad N = \dfrac{\pi}{\lambda}\left(\dfrac{1}{\rho}+\dfrac{1}{\rho_1}\right).$

Thus

$$s_0 = A'\left\{\sin\frac{2\pi t'}{T}\int\cos(Lx+Mx^2+Nz^2)\,dS-\right.$$
$$\left.-\cos\frac{2\pi t'}{T}\int\sin(Lx+Mx^2+Nz^2)\,dS\right\},$$

and the intensity is given by $J = A'^2(C^2+S^2)$, where

$$C = \iint \cos(Lx+Mx^2+Nz^2)\,dxdz,$$

$$S = \iint \sin(Lx+Mx^2+Nz^2)\,dxdz.$$

The integration with respect to z gives no trouble, and on carrying it out we are left with

$$J = \frac{A'^2\pi}{N}(c^2+s^2), \tag{3}$$

where

$$c = \int \cos(Lx+Mx^2)\,dx, \qquad s = \int \sin(Lx+Mx^2)\,dx, \tag{4}$$

and the integrals are still to be taken over the reflecting parts of the grating.

3. The form of the solution of the plane-grating problem is obtained immediately by letting $R \to \infty$, and this shows the exact nature of the correspondence between the plane and concave gratings which was mentioned above. It is interesting to see what happens to the focal curve for a plane grating. The equation $M = 0$ gives $\cos^2 u/\rho + \cos^2 i/\rho_1 = 0$, so that ρ, ρ_1 have opposite signs (unless, of course, both are infinite). Thus the only diffraction problems of a plane grating which are soluble by 'Fraunhofer integrals' are those in which the incident light is convergent, or in which diverging diffracted rays are gathered by a lens. These cases are well known, but the present theory unites into one calculation all problems soluble by Fraunhofer integrals, whether for plane or concave gratings.